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Editor

EDITORIAL

This issue contains problems and solutions from the Mathematics competitions held during 2015.

There are some sections of Mathematics teachers who question the need for Mathematics talent competitions. Their argument is that many students who attempt such competitions feel discouraged that they could not solve many problems (some times not even one!). This also makes them decide that "they are not good enough".

In my opinion, there is immense value in participating in such competitions and effort spent in pondering over such problems is very rewarding. At a minimum, the student gets an exposure to interesting mathematics and thrill of discovery. These competitions also strengthen the problem solving skill. One should take these competitions with a sportive spirit – winning is not the aim – the experience gained is the grand prize. I can assure you that in my 35+ years of professional life (I am not a teacher of Mathematics – I work with computers!) not a single day passes without thanking the problem solving skill Mathematics taught me.

I am sure you will enjoy the problems and solutions in this issue. Perhaps you may find an alternate (and more beautiful) solution to some of them. A good problem is one that allows several solutions and a great problem is one that could be done entirely in one's mind with an insightful observation.

Let me close by presenting the following great problem from the American Invitational Mathematics Examination 2016. This can be solved without making any computations by observing a small pattern!
Let $P(x) = 1 - \frac{1}{3}x + \frac{1}{6}x^2$ and

$$Q(x) = P(x)P(x^3)P(x^5)P(x^7)P(x^9) = \sum_{i=0}^{50} a_i x^i$$

If $\sum_{i=0}^{50} |a_i| = \frac{m}{n}$, where m, n are relatively prime positive integers, find $m + n$.

Answer is 275.

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SCREENING TEST – GAUSS CONTEST
NMTC at PRIMARY LEVEL
V & VI Standards

PART – A

1. A three digit number is divisible by 35. The greatest such number has in its tenth place the digit

   A. 4 B. 7 C. 9 D. 8

   Solution The largest three digit number is 999 and it leaves a remainder 19 when divided by 35. Thus the largest three digit number divisible by 35 is $999 - 19 = 980$. Hence the number in the tens place is 8. Answer is D.

2. When 2^{2015} is completely calculated, the units place of the number obtained is

   A. 2 B. 4 C. 6 D. 8

   Solution We have

   $$2^1 = 2,\ 2^2 = 4,\ 2^3 = 8,\ 2^4 = 16,\ 2^5 = 32,\ 2^6 = 64, \ldots$$

   Hence the units digits of powers of 2 are 2, 4, 8, 6, 2, 4, 8, 6, ... Hence $2^4, 2^8, 2^{12}, \ldots, 2^{4n}$ have units digit as 2, $2^5, 2^9, 2^{13}, \ldots, 2^{4n+1}$ have units digit 4, $2^6, 2^{10}, 2^{14}, \ldots, 2^{4n+2}$ have units digit 8 and $2^7, 2^{11}, 2^{15}, \ldots, 2^{4n+3}$ have units digit 6. Since $2015 = 4 \times 503 + 3$, it follows that the units digit of 2^{2015} is 6. Answer is C.

3. The smallest integer that is an integer multiple of $4\frac{1}{2}$, 3, $10\frac{1}{2}$ is

   A. 62 B. 18 C. 63 D. 64

   Solution The given fractions are $\frac{9}{2}$, 3, $\frac{21}{2}$. Since the

least common multiple of $9, 3,$ and 21 is 63, the required integer is 63. Answer is C.

4. The ratio of the money with Samrud and Saket is $7:15$ and that with Saket and Viswa is $7:16$. If Samrud has Rs 490, the amount of money in Rupees Viswa has is
A. 2000 B. 4900 C. 2400 D. 2015
Solution $Samrud : Saket = 7 : 15 = 49 : 105$ and $Saket : Viswa = 7 : 16 = 105 : 240$. Thus
$$Samrud : Saket : Viswa = 49 : 105 : 240$$
Since Samrud has Rs 490, the amount of money with Viswa is $\frac{490}{49} \times 240 = 2400$. Answer is thus C.

5. Two numbers are respectively 26% and 5% more than a third number. What percent is the first of the second?
A. 80% B. 120% C. 93% D. 75%
Solution Let the third number be X. The first number a is $1.26 \times X$ and the second number b is $1.05 \times X$. Thus $\frac{a}{b} = \frac{1.26 \times X}{1.05 \times X} = \frac{126}{105} = 1.2$. Thus the first number is 120% of the second number. Answer is B.

6. The average age of 24 students and their class teacher is 16 years. If the class teacher's age is excluded, the average age reduces by 1 year. The age of the class teacher in years is
A. 40 B. 45 C. 50 D. 55
Solution The total age of the students and their class teacher is $16 \times 25 = 400$. The total age of the students alone is $15 \times 24 = 360$. Thus the age of the class teacher is $400 - 360 = 40$ years. Answer A.

7. Mahadevan told his grand daughter "I am 66 years old if I do not count the Sundays". The actual age of Mahadevan

is

A. 77 B. 79 C. 83 D. 88

Solution Since he ignores Sundays, when he was seven days old, he would have said he is only 6 days old, when he is 7 years old, he would have said he is 6 years old etc. Thus when he says he is 66 years old, his actual age is $\frac{66}{6} \times 7 = 77$ years. Answer is A.

8. The length of a rectangle is 9 times its width. The ratio of its perimeter to the perimeter of a square of same area is

A. $5:4$ B. $6:5$ C. $5:3$ D. $7:5$

Solution Let the width of the rectangle be x. Then the length is $9x$ and its area is $9x^2$. The side of the square with the same area is $3x$. The perimeter of the rectangle is $2(9x+x) = 20x$ and the perimeter of the square is $4(3x) = 12x$. Thus the required ratio is $20:12 = 5:3$. Answer C.

9. $ABCD$ is a square of side 1 cm and O is the point of intersection of the diagonals. P is the midpoint of OB. AP^2 equals

A. $\dfrac{3}{8}$ B. $\dfrac{3}{4}$ C. $\dfrac{3}{5}$ D. $\dfrac{5}{8}$

Solution By Pythagoras theorem, $BD^2 = AB^2 + AD^2 = 2$. Hence $BD = \sqrt{2}$. Note also that $AC = BD$ and hence $AO = \frac{1}{2}AC = \frac{1}{\sqrt{2}}$. Also, $OP = \frac{1}{4}BD = \frac{1}{2\sqrt{2}}$. Thus $AP^2 = AO^2 + OP^2 = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$. Answer D.

10. The average of 10 consecutive odd numbers is 120. What is the average of the five smallest numbers among them?

A. 100 B. 105 C. 110 D. 115

Solution Let the numbers be $a, a+2, \ldots, a+18$. The

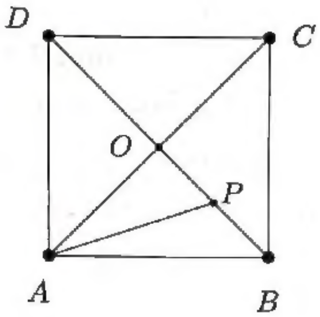


Figure for Q 9.

sum of these is $5 \times (a + a + 18) = 10 \times (a + 9)$. Since their average is 120, we have $a + 9 = 120$ and hence $a = 111$. Thus the five smallest of these numbers are $111, 113, 115, 117, 119$ and their average is 115. Answer D.

11. abc is a three digit number where a, b, c are digits. How many integers are there such that $a \times b \times c = 12$?

    A. 12 B. 6 C. 4 D. 15

    Solution We can write 12 as a product of three numbers in the following ways: $(12, 1, 1), (6, 2, 1), (4, 3, 1), (3, 2, 2)$. Since 12 is not a digit, $(12, 1, 1)$ is not possible. When the digits are $(6, 2, 1)$ we have six numbers: $621, 612, 261, 216, 126, 162$. When the digits are $(4, 3, 1)$ we have the six numbers: $431, 413, 341, 314, 134, 143$. When the digits are $(3, 2, 2)$ we have three numbers: $322, 223, 232$. Thus there are 15 such numbers. Answer is D.

12. *Slok* is a primary school kid. He counted the number of Sundays occurring in 45 consecutive days. He was very happy he got the maximum Sundays possible. This

maximum number is

A. 6 B. 7 C. 8 D. 5

Since there are 6 whole weeks and 3 days in 45 consecutive days, the maximum number of Sundays is at most 7. Seven Sundays will occur if the first day is a Sunday. Then we have the following Sundays: 1, 8, 15, 22, 29, 36, 43. Answer is B.

13. The value of

$$\frac{50}{72}+\frac{50}{90}+\frac{50}{110}+\frac{50}{132}+\cdots+\frac{50}{9900}$$

is

A. $\frac{23}{4}$ B. $\frac{32}{7}$ C. $\frac{1}{2015}$ D. $\frac{55}{27}$

Solution

$$\begin{aligned}
&\frac{50}{72}+\frac{50}{90}+\frac{50}{110}+\frac{50}{132}+\cdots+\frac{50}{9900}\\
&\quad=50\left(\frac{1}{8\times 9}+\frac{1}{9\times 10}+\frac{1}{10\times 11}+\cdots+\frac{1}{99\times 100}\right)\\
&\quad=50\left(\left(\frac{1}{8}-\frac{1}{9}\right)+\left(\frac{1}{9}-\frac{1}{10}\right)+\cdots\left(\frac{1}{99}-\frac{1}{100}\right)\right)\\
&\quad=50\left(\frac{1}{8}-\frac{1}{100}\right)\\
&\quad=\frac{50}{8}-\frac{50}{100}=\frac{25}{4}-\frac{1}{2}=\frac{23}{4}
\end{aligned}$$

Answer is A.

14. In the following figure, the distance between any two adjacent dots horizontally or vertically is 1 unit. If A is the area of the shaded region in Figure 1 and B is the area of the shaded region in Figure 2, the ratio $A:B$ is

A. $4:3$ B. $5:1$ C. $4:1$ D. $6:1$

Solution In Figure 1, the region consists of a rectangle with sides 2 units by 1 unit, one square with side 1

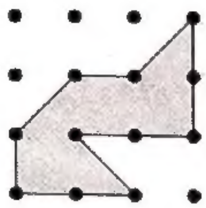


Figure 1

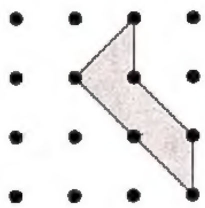


Figure 2

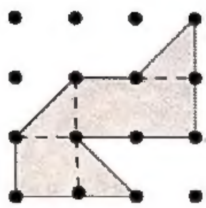


Figure 1

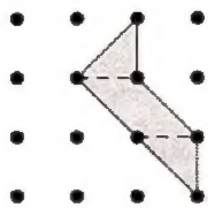


Figure 2

unit three right angled triangles with non-hypotenuse sides equal to 1 unit. Thus the area of the region is $A = 2\times 1 + 1\times 1 + 3\times \frac{1}{2}\times 1\times 1 = \frac{9}{2}$ square units.
Similarly, the region in Figure 2 contains two right angled triangles with non-hypotenuse sides equal to 1 unit and a parallelogram with base 1 unit and height 1 unit. Thus the area is $B = 2\times\frac{1}{2}\times 1\times 1 + 1\times 1 = 2$ square units.
Thus $A : B = \frac{9}{2} : 2 = 9 : 4$.
Remark In the above, we have split the region into triangles, squares and parallelograms. Consider the more complicated polygonal region in Figure 3: We can compute the area using the same technique but there is a beautiful theorem called Pick's Theorem that gives the formula for such areas. If B is the number of dots on the boundary of the polygon and I the number of dots

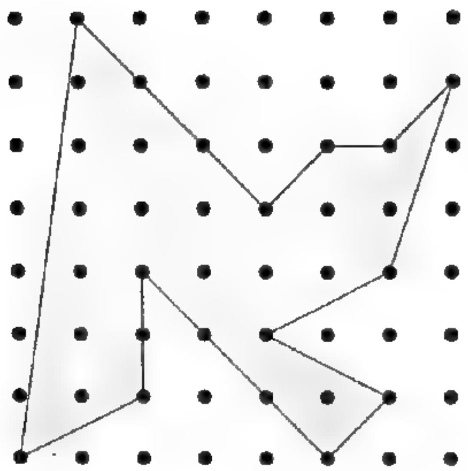


Figure 3

in the interior of the polygon, then the area is given by $I + \frac{B}{2} - 1$.

For example in Figure 1 in the question, we had $I = 0$ and $B = 11$ and hence the area by Pick's theorem is $0 + \frac{11}{2} - 1 = \frac{9}{2}$, the same as what we obtained.

In Figure 3, we have $I = 14$ and $B = 18$ and hence the area is $14 + 9 - 1 = 22$ square units. You can check the answer by splitting the region into triangles, rectangles and parallelograms.

15. Which one among the choices given below is true for the set of five natural numbers $24, 25, 26, 27, 28$?

    A. When we add $3, 4, 5, 6, 7$ respectively to the numbers, we get 5 prime numbers

    B. When 5 is added to 24, 6 is subtracted from 25, 7 is added to 26, 8 is subtracted from 27 and 9 is added to 28 we get a set of 5 prime numbers

    C. All the five consecutive numbers are composite

D. When 1 is added to each number we get a set of prime numbers

Solution Clearly C is true. It is easy to see that the other choices are false.

PART – B

16. If 75 is written as the sum of 10 consecutive natural numbers, the maximum of the numbers is

A. 15 B. 25 C. 34 D. 12

Solution If a is the largest number, then we must have

$$\begin{aligned} 75 &= a + (a-1) + \cdots + (a-9) \\ &= 10a - (1 + 2 + \cdots + 9) \\ &= 10a - 45 \end{aligned}$$

Thus $a = 12$. Answer is D.

17. The number of four digit numbers greater than 2000 which contain the digits of 2015 (without repetition of digits) is ———.

Solution We need to find the number of four digit numbers that have only the digits $2, 0, 1, 5$ and more than 2000. Since neither 0 nor 1 can be the first digit, we can fill the first digit in 2 ways (with 2, 5), second digit in 3 ways (digits not used in the first digit), third digit in 2 ways and fourth digit in 1 way. Hence there are $2 \times 3 \times 2 \times 1 = 12$ four digit numbers that satisfy the required properties.

18. In a box there are green, red and blue beads. The number of beads which are not green is 9, number of beads which are not red is 8 and the number of beads which are not blue is 7. Total number of beads in the box is ———.

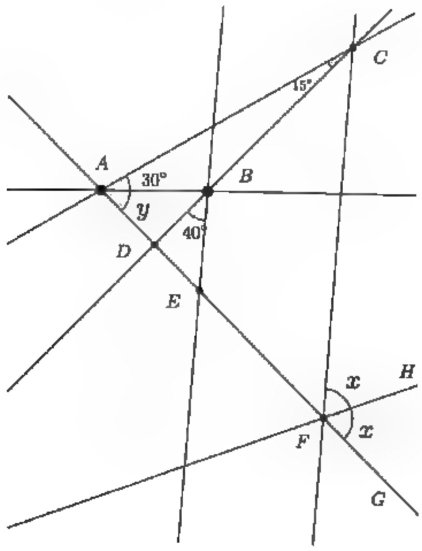
Solution Let G, R, B be the numbers of green, red and blue beads respectively. We have $R+B=9$, $G+B=8, G+R=7$. Adding, we get $2(G+R+B)=24$ and hence $R+G+B=12$. Thus the total number of beads in the box is 12.

19. A sequence of numbers starts with $1, 2, 3$ and every number of the sequence from the fourth term is the sum of the previous three numbers. The tenth number in the sequence is ——.

    Solution The sequence is $1, 2, 3, 6, 11, 20, 37, 68, 125, 230$ and hence the tenth number is 230.

20. In the adjoining figure, ABC is a triangle. AD is perpendicular to CB produced. BE is parallel to CF and FH bisects $\angle CFG$. The value of $x+y$ is ——.

    Solution $\angle ABD = 30° + 15° = 45°$. Since $\angle BDA =$



$90°$, it follows that $y = 45°$. Now, since $\angle BDE = 90°$, $\angle BED = 90° - 40° = 50° = \angle CFE$. Thus $x = \dfrac{180° - \angle CFE}{2} = 65°$ and $x+y = 45°+65° = 110°$.

SCREENING TEST – KAPREKAR CONTEST
NMTC at SUB JUNIOR LEVEL
VII & VIII STANDARDS

PART - A

1. The ratio of the angles of a quadrilateral are in the ratio $7:9:10:10$. Then

   A. One angle of the quadrilateral is greater than $120°$

   B. Only one angle of the quadrilateral is $90°$

   C. The sum of some two angles of the quadrilateral is $100°$

   D. There are exactly two right angles as interior angles

   Solution Since the sum of the angles of the quadrilateral is $360°$, it follows that the angles are $70°, 90°, 100°, 100°$. Hence we have exactly one angle that equals $90°$ and the answer is B.

2. The sum of three different integers is 1 and their product is 36. Then

   A. All of them are positive

   B. Only one is negative

   C. Exactly two are negative

   D. All the three are negative

   Solution Let a, b, c be the three integers. Since $a \times b \times c = 36$, either all are positive or exactly two of them are negative. If all are positive, then $a, b, c \geq 1$ gives $a + b + c \geq 3$ but we are given that the sum of the integers is 1. Thus exactly two of them must be negative and the answer is C.

3. The value of $\underbrace{2^{2015}+2^{2015}+\cdots+2^{2015}}_{256\,terms}$ divided by 2^{2015} is

A. 256 B. 2^{73} C. 2^{2015} D. 2015

Solution

$$\underbrace{2^{2015}+2^{2015}+\cdots+2^{2015}}_{256\,terms}=256\times 2^{2015}$$

Hence the value of $\underbrace{2^{2015}+2^{2015}+\cdots+2^{2015}}_{256\,terms}$ divided by 2^{2015} is 256 and the answer is A.

4. For a, b, define

$$a*b=\frac{ab+ba}{a+b}$$

where by ab we mean writing b after a and interpret the resulting sequence as a decimal number. For example,

$$155*60-\frac{15560+60155}{155+60}$$

If $a=2015$ and $b=5$, $a*b$ lies between

A. 35 and 36 B. 37 and 38 C. 51 and 52 D. 53 and 54

Solution

$$2015*5=\frac{20155+52015}{2015+5}=\frac{72170}{2020}=35.72$$

Hence the answer is A.

5. The 2015^{th} letter of the sequence

$$ABCDEDCBAABCDEDCBA\ldots$$

is ——.

Solution Here the sequence of letters $ABCDEDCBA$ is repeated. Since there are 9 letters in this sequence, in the first 2007 letters we will have this sequence of 9 letters repeated 223 times. Now we have 8 more letters to fill and hence the 2015^{th} letter is same as the eighth letter in the sequence $ABCDEDCBA$ and thus is B.

6. n is a natural number. The number of possible reminders when n^2 is divided by 7 is
A. 2 B. 3 C. 4 D. 5
Solution We can write $n = i+7j$ where i, j are integers and $0 \leq i < 7$. We have $n^2 = i^2 + 14ij + 49j^2$ and hence n^2 leaves the same reminder as i^2 when divided by 7. Thus it is sufficient to look at the reminders of $0^2, 1^2, 2^2, 3^2, 4^2, 5^2, 6^2$ when divided by 7. These are $0, 1, 4, 2, 2, 4, 1$ and hence the possible reminders are $0, 1, 2, 4$. The answer is C.

7. The ratio of two numbers is $7 : 9$. If each number is decreased by 2, the ratio becomes $3 : 4$. The sum of the two numbers is A. 23 B. 32 C. 48 D. 12
Solution Let the numbers be x, y. We have $\dfrac{x}{y} = \dfrac{7}{9}$ and $\dfrac{x-2}{y-2} = \dfrac{3}{4}$. Thus $7y = 9x$ and $3(y-2) = 4(x-2)$. Solving, we get $x = 14,\ y = 18$. Thus $x + y = 32$ and answer is B.

8. The speeds of two runners are respectively $15\,km/hr$ and $16\,km/hr$. To cover a distance of $d\,kms$ one takes 16 minutes more than the other. Then d in kms is
A. 32 B. 48 C. 64 D. 128
Solution To cover $d\ \ kms$, the first runner will take $\frac{d}{15}$ hours and the second runner will take $\frac{d}{16}$ hours. We are given that $\dfrac{d}{15} = \dfrac{d}{16} + \dfrac{16}{60}$. Note that since the speed is in km/hr we have converted 16 minutes into hours. Solving for d, we obtain

$$\frac{d}{15} - \frac{d}{16} = \frac{d}{240} = \frac{16}{60}$$

and we obtain $d = 64\,kms$. Answer is C.

9. In the sum $3 + 33 + 333 + 3333 + 33333 + \cdots$ where there are 2015 terms, the number formed by taking the last four

digits in order is

A. 6365 B. 6255 C. 6465 D. 6565

Solution Since we are interested only in the last four digits, it is enough to evaluate the last four digits of the sum $3 + 33 + 333 + 3333 + 33333 + \cdots$ where every term after the third term is 3333. Thus the sum is $369 + 2012 \times 3333 = 6706365$. Thus the number formed by the last four digits of the sum is 6365. Answer is A.

10. $a\%$ of the quantity P is added to P. To the increased quantity, $b\%$ of the increased quantity is added. $c\%$ of the result is added to the result and the final quantity is Q. Then P is

A. $\dfrac{Q \times 100 \times 100 \times 100}{(a + b + c)}$

B. $\dfrac{Q}{100(a + b + c)}$

C. $\dfrac{Q \times 100 \times 100 \times 100}{(100 + a) + (100 + b) + (100 + c)}$

D. $\dfrac{Q \times 100 \times 100 \times 100}{(100 - a) + (100 - b) + (100 - c)}$

Solution

If P is increased by $a\%$, we obtain $P \times \left(1 + \frac{a}{100}\right)$. Hence we have

$$Q = P \times \left(1 + \frac{a}{100}\right) \times \left(1 + \frac{b}{100}\right) \times \left(1 + \frac{c}{100}\right)$$
$$= \frac{P \times (100 + a) \times (100 + b) \times (100 + c)}{100 \times 100 \times 100}$$

Hence

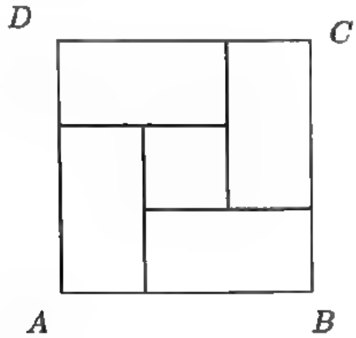
$$P = \frac{Q \times 100 \times 100 \times 100}{(100 + a) + (100 + b) + (100 + c)}$$

Answer is C.

11. $ABCD$ is a square with area 64 sq cms. The center square has area 16 sq cms. The remaining are four congruent rectangles. The ratio $\frac{length}{breadth}$ of the rectangle is

A. 2 B. 3 C. 4 D. 5

Solution The length of the side of the outer square is 8



cms. If a, b are the length and breadth of the rectangles, then we have $a+b=8$ and the length of the side of the inner square is $a-2b$. Thus $a-2b=4$. Solving for a, b we get $a=\frac{20}{3}$ and $b=\frac{4}{3}$. Thus $\frac{length}{breadth}=\frac{20/3}{4/3}=5$ and the answer is D.

12. If $3^a+3^b=756, 7^a+2^c=375$ and $5^a+3=128$, the value of $a+b+c$ is

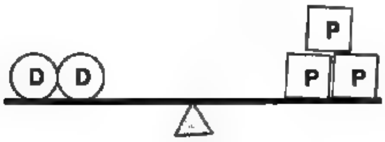
A. 12 B. 14 C. 18 D. 20

Solution Since $5^a+3=128$, it follows that $5^a=125=5^3$ and hence $a=3$. From $7^a+2^c=375$, we obtain $2^c=375-7^3=32$ and consequently, $c=5$. Finally, from $3^a+3^b=756$ we get $3^b=756-3^3=729=3^6$ and $b=6$. Thus $a+b+c=3+6+5=14$ and the answer is B.

13. There are four types of dolls called *Dingle* (D), *Pingle* (P), *Jingle* (J) and *Mingle* (M). All toys of the same category have the same weight. Toys of different category have different weights. They balance as shown below. How

many *Jingles* will balance a *Mingle*?

A. 2 B. 3 C. 4 D. 5



Solution If d, p, j, m respectively denote the weights of the toys, we have $2d = 3p$, $d = j + m$ and $p + j = m$. Hence

$$m = p + j = \frac{2d}{3} + j = \frac{2}{3}(j + m) + j = \frac{5}{3}j + \frac{2}{3}m$$

and $\frac{1}{3}m = \frac{5}{3}j$. Thus $m = 5j$ and the answer is D.

14. A student has to score 30% marks to get through an examination. If he gets 30 marks and fails by 30 marks, the maximum marks set for the examination is

A. 90 B. 200 C. 250 D. 125

Solution Since he fails by 30 marks when he gets 30 marks, the required marks for pass is 60 marks. This is 30% of the total marks. Hence the total marks for the examination is $\frac{60}{30} \times 100 = 200$ marks. Answer is B.

15. a, b, c, d are real numbers such that $1015 \leq a \leq 2015$, $3015 \leq b \leq 4015$, $5015 \leq c \leq 6015$ and $7015 \leq d \leq 8015$. The maximum value of $\dfrac{c+d}{a+b}$ is

A. $\dfrac{1403}{403}$ B. $\dfrac{1402}{403}$ C. $\dfrac{1401}{403}$ D. 2015

Solution Maximum of $c+d$ happens when $c = 6015$ and $d = 8015$. The minimum of $a+b$ happens when $a = 1015$ and $b = 3015$. Thus the maximum value of $\dfrac{c+d}{a+b}$ is

$$\frac{6015+8015}{1015+3015} = \frac{14030}{4030} = \frac{1403}{403}$$

and the answer is A.

16. A black and white photograph is 70% black and 30% white. It is enlarged three times. The percentage of white in the enlargement is

A. 90% B. $66\frac{2}{3}\%$ C. $33\frac{1}{2}\%$ D. 30%

Solution Since in an enlargement, every black dot and white dot is enlarged, the ratio should remain the same. Thus the answer is D.

17. The units digit in the product

$$(5+1)(5^2+1)(5^3+1)\cdots(5^{2015}+1)$$

is

A. 9 B. 8 C. 6 D. 4

Solution Since any power of 5 has 5 as the units digit, for any n, the units digit of $5^n + 1$ is 6. Thus we want the units digit of

$$\underbrace{6 \times 6 \times \cdots \times 6}_{2015\,terms}$$

Again, all powers of 6 end with 6. Hence the units digit of the product given is 6. Answer is C.

18. If the product of the digits of a 4 digit number is 75, the sum of its digits is

A. 12 B. 13 C. 14 D. 15

Solution Since $75 = 1 \times 3 \times 5 \times 5$, it follows that the digits must be $1, 3, 5, 5$ in some order. Thus the sum of the digits is $1 + 3 + 5 + 5 = 14$ and the answer is C.

19. The hypotenuse c and one side a of a right angled triangle are consecutive integers. The square of the third side is

A. $c - a$ B. ca C. $c + a$ D. c/a

Solution Given $c = a + 1$. Thus the square of the other side b is given by $b^2 = c^2 - a^2 = (c - a)(c + a) = c + a$. Answer is C.

20. The fraction $\dfrac{2121212121210}{1121212121211}$ when reduced to its simplest form is

A. $\dfrac{73}{70}$ B. $\dfrac{37}{7}$ C. $\dfrac{70}{37}$ D. $\dfrac{70}{13}$

Solution

$$\begin{aligned} \frac{2121212121210}{1121212121211} &= \frac{3 \times 707070707070}{3 \times 373737373737} \\ &= \frac{70 \times 10101010101}{37 \times 10101010101} \\ &= \frac{70}{37} \end{aligned}$$

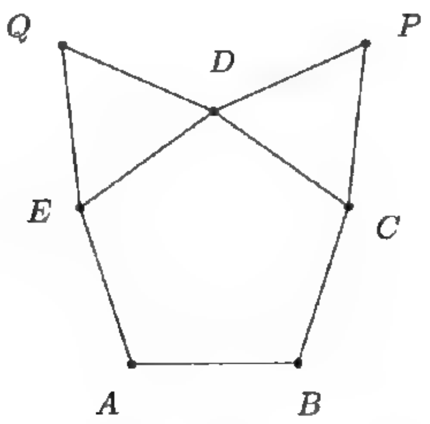
Answer is C.

PART - B

21. The number of integers in $1, 2, 3, \ldots, 2015$ that are perfect squares and perfect cubes is ———.

Solution If a number is a perfect square and a perfect cube, then it is necessarily a sixth power of an integer. Hence we need to count the integers $1^6, 2^6, 3^6 \ldots n^6$ where $n^6 \leq 2015$. Since $3^6 = 729$ and $4^6 = 4096$, it follows that $1^6, 2^6, 3^6$ are the only sixth powers in the list of numbers given. Thus the answer is 3.

22. $ABCDE$ is a regular pentagon. CDP and EDQ are equilateral triangles. The measure of $\angle QDP$ is ———.
Solution The size of each interior angle in a regular



pentagon is $108°$. Since PDC and QDE are equilateral triangles, we have

$$\angle PDQ = 360° - 60° - 60° - 108° = 132°$$

23. The value of $1 - 2 + 3 - 4 + 5 - \cdots + 2015$ is ———.
Solution We have

$$\begin{aligned} S &= 1 \quad -2+3 \quad -4 - \cdots + 2015 \\ &= 2015 \quad -2014 + 2013 \quad -2012 + \cdots + 1 \end{aligned}$$

Hence

$$\begin{aligned} 2S &= (2016 - 2016) + \cdots + (2016 - 2016) + 2016 \\ &= 0 + 0 + \cdots + 0 + 2016 = 2016 \end{aligned}$$

Thus $S = 1008$.
We can also see this as follows:

$$\begin{aligned} S &= (1-2)+(3-4)+\cdots+(2013-2014)+2015 \\ &= \underbrace{-1-1-1-\cdots-1}_{1007\,terms}+2015 \\ &= 1008 \end{aligned}$$

24. Using the digits of the number 2015, four digit numbers of different digits are formed. The number of such numbers greater than 2000 and less than 6000 is ———.
Solution We need to find the number of four digit numbers that have only the digits $2, 0, 1, 5$, more than 2000 and less than 6000. Clearly all four digit numbers using only these four digits are less than 6000. Since neither 0 nor 1 can be the first digit, we can fill the first digit in 2 ways (with 2, 5), second digit in 3 ways (digits not used in the first digit), third digit in 2 ways and fourth digit in 1 way. Hence there are $2 \times 3 \times 2 \times 1 = 12$ four digit numbers that satisfy the required properties.

25. *Samrud* got an average mark 85 in his first 8 tests and an average mark 81 in the first 9 tests. His mark in the 9^{th} test is ———.
Solution His total marks in the first 8 tests is $8 \times 85 = 680$ and his total marks in the first 9 tests is $9 \times 81 = 729$. Hence his marks in the 9^{th} test is $729 - 680 = 49$.

26. The remainder when 20150020150002015002015 is divided by 3 is ———.
Solution The remainder when a number is divided by 3 is the same as the remainder when the sum of the digits of the number is divided by 3. The sum of the digits of the given number is 32 and hence the remainder is 2.

27. If

$$\frac{p}{q} = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \frac{1}{5}}}}$$

where p, q have no common factors, then $p+q =$ ———.

Solution

$$\frac{p}{q} = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \frac{1}{5}}}} = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \frac{5}{21}}}$$

$$= 1 + \cfrac{1}{2 + \frac{21}{68}} = 1 + \frac{68}{157} = \frac{225}{157}$$

Hence $p + q = 225 + 157 = 382$.

28. If

$$\frac{p}{q} = 1 + \cfrac{5}{1 + \cfrac{4}{1 + \cfrac{3}{1 + \frac{1}{2}}}}$$

where p, q have no common factors, then $p+q =$ ———.

Solution It is easy to see that $\frac{p}{q} = \frac{22}{7}$. Thus $p+q = 29$.

29. In the Figure for Q 29, $ABCD$ is a rectangle. $AD -$ 2, $AB = 1$, AE is the arc of the circle with center D. The length BE is equal to ———.

Solution Join DE. In the right angled triangle DEC, $DE = 2$, $DC = 1$ and hence $EC = \sqrt{DE^2 - DC^2} = \sqrt{3}$. Since $BC = 2$, it follows that $BE = 2 - \sqrt{3}$.

30. In the Figure for Q 30, the squares have are $1\,cm^2$ the rectangles have area $2\,cm^2$. The number of squares with

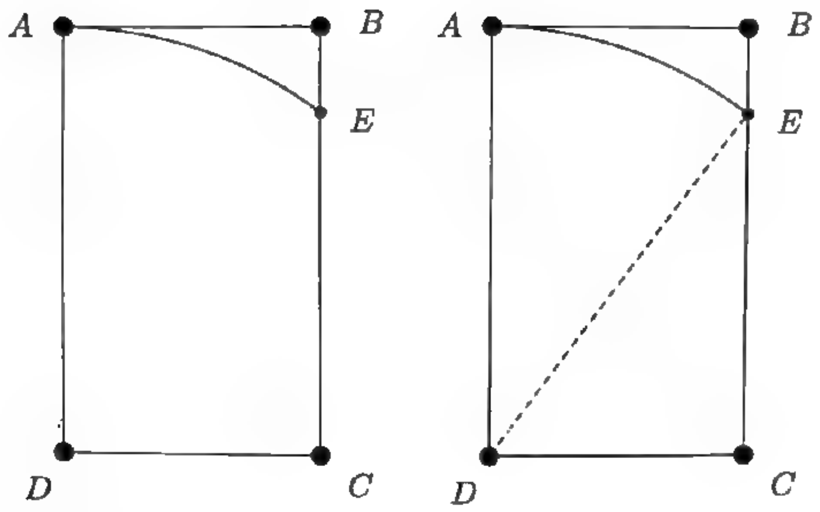


Figure for Q 29

different dimensions in the figure is ———.

Solution From the figure it is clear that the individual

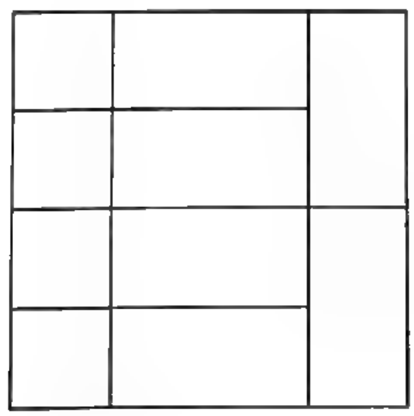


Figure for Q 30

rectangles are of size 1×2. There are four squares of size 1×1, three squares with dimension 2×2, two squares of dimension 3×3 and one square with dimension 4×4.

SCREENING TEST – BHASKARA CONTEST
NMTC at JUNIOR LEVEL
IX & X Standards

PART - A

1. The number of real solutions of the equation

$$\sqrt[3]{x-1}+\sqrt[3]{x-3}+\sqrt[3]{x-5}=0$$

is A. 0 B. 1 C. 2 D. 3

Solution We know that if $a+b+c=0$, then $a^3+b^3+c^3=3abc$. Thus we have

$$(x-1)+(x-3)+(x-5)=3\sqrt[3]{(x-1)(x-3)(x-5)}$$

Hence

$$\begin{aligned} 3(x-3) &= 3\sqrt[3]{(x-1)(x-3)(x-5)} \\ \Rightarrow \quad \sqrt[3]{x-3}\left(\sqrt[3]{(x-3)^2}-\sqrt[3]{(x-1)(x-5)}\right) &= 0 \end{aligned}$$

Hence either $x-3=0$ or

$$\sqrt[3]{(x-3)^2}-\sqrt[3]{(x-1)(x-5)}=0$$

This gives

$$x^2-6x+9=x^2-6x+5 \Rightarrow 9=5$$

an impossibility. Thus $x=3$ is the only real solution and the answer is B.

2. A merchant has 100 kg of sugar, part of which he sells at 7% profit and the rest at 17% profit. He gains 10% on the whole. The amount in kg of sugar he sold at 7% profit is
A. 60 B. 50 C. 80 D. 70
Solution Suppose that he sells m kg for 7% profit and

$100 - m$ kg at 17% profit. If C is the cost price of a kg of sugar, we have

$$m \times 1.07 \times C + (100 - m) \times 1.17 \times C = 100 \times 1.1 \times C$$

Thus

$$117 - 0.1m = 110$$

and $m = 70$. Answer is D.

3. Mahadevan was asked what is $\frac{16}{17}$ of a certain fraction. By mistake he divided the fraction by $\frac{16}{17}$ and got an answer that exceeded the correct answer by $\frac{33}{340}$. The correct answer is

   A. $\frac{60}{87}$ B. $\frac{62}{85}$ C. $\frac{64}{85}$ D. $\frac{67}{85}$

   Solution Let the fraction be F. Given that

$$\frac{F}{\frac{16}{17}} - F \times \frac{16}{17} = \frac{33}{340}$$

   Simplifying, we get $F = \frac{4}{5}$. Thus the correct answer is

$$\frac{4}{5} \times \frac{16}{17} = \frac{64}{85}$$

   Answer is C.

4. When $a = 2015$ and $b = 2016$, the value of

$$\frac{a\sqrt{a} + b\sqrt{b}}{(\sqrt{a} + \sqrt{b})(a - b)} + \frac{2\sqrt{b}}{\sqrt{a} + \sqrt{b}} - \frac{\sqrt{ab}}{a - b}$$

   is A. 0 B. 1 C. $(2015)^2$ D. $\sqrt{2016}$

Solution

$$\begin{aligned}
&\frac{a\sqrt{a}+b\sqrt{b}}{(\sqrt{a}+\sqrt{b})(a-b)}+\frac{2\sqrt{b}}{\sqrt{a}+\sqrt{b}}-\frac{\sqrt{ab}}{a-b}\\
&=\frac{a\sqrt{a}+b\sqrt{b}+2\sqrt{b}(a-b)-\sqrt{ab}(\sqrt{a}+\sqrt{b})}{(\sqrt{a}+\sqrt{b})(a-b)}\\
&=\frac{a\sqrt{a}-b\sqrt{b}+a\sqrt{b}-b\sqrt{a}}{(\sqrt{a}+\sqrt{b})(a-b)}\\
&=\frac{(\sqrt{a}+\sqrt{b})(a-b)}{(\sqrt{a}+\sqrt{b})(a-b)}\\
&=1
\end{aligned}$$

Answer is B.

5. An arithmetic progression has positive terms. The ratio of the difference of the 4th and 8th term to the 15th term is $\frac{4}{15}$ and the square of the difference of the 4th and the 1st term is 225. Which term of this series equals 2015?

   A. 225 B. 404 C. 403 D. 410

   Solution Let the first term be a and common difference d. We have

$$\frac{(a+7d)-(a+3d)}{a+14d}=\frac{4}{15}$$
$$(a+3d-a)^2=225$$

   Hence $d = 5$. It follows that $a = 5$. Hence if $a+(n-1)d=2015$, we have $n=403$. Answer is C.

6. The number of values of x which satisfy the equation $5^x \times \sqrt[x]{8^{x-1}} = 500$ is

   A. 1 B. 2 C. 3 D. 0

   Solution The given equation can be written as

$$5^x \times 2^{3(x-1)/x} = 5^3 \times 2^2 \tag{1}$$

   Taking logarithms in (1), we get

$$x\log 5+\frac{3x-3}{x}\log 2=3\log 5+2\log 2 \tag{2}$$

Simplifying (2), we get

$$x^2 \log 5 + x \log\left(\frac{2}{125}\right) - 3\log 2 = 0 \qquad (3)$$

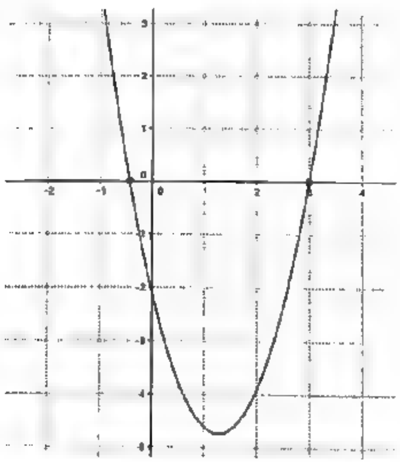
The solutions of (3) are given by

$$\frac{-\log\left(\frac{2}{125}\right) \pm \sqrt{\left(\log\left(\frac{2}{125}\right)\right)^2 + 12 \cdot \log 2 \cdot \log 5}}{2\log 5}$$

This simplifies to

$$\frac{-(\log 2 - 3\log 5) \pm (\log 2 + 3\log 5)}{2\log 5}.$$

Taking the positive sign, we get $x = 3$ and taking the negative sign, we get the solution $x = -\dfrac{\log 2}{\log 5}$. Thus the given equation has two solutions and the answer is B. The graph of the quadratic (3) is shown below.



7. A number when divided by 899 gives a remainder 63. The remainder when this number is divided by 29 is
A. 6 B. 7 C. 8 D. 5
Solution Note that $899 = 31 \times 29$. If N leaves a

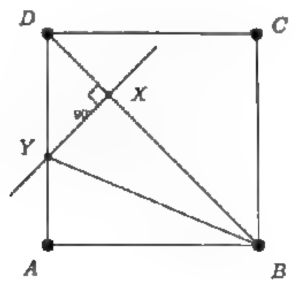
remainder 63 when divided by 899, then $N - 63 = 899k$ for some integer k. Hence

$$N = 63 + 899k = 5 + 2 \times 29 + 31k \times 29$$

and N will leave a remainder 5 when divided by 29.

8. A train leaves a station 1 hour before the scheduled time. The driver decreases the speed by 4 km per hour. At the next station 120 kms away, the train reached on scheduled time. The original speed of the train in km per hour is
A. 24 B. 36 C. 18 D. 22
Solution If the original speed was s kms per hour, we have $\frac{120}{s} = \frac{120}{s-4} - 1$. Solving for s, we obtain $s = 24$ or -20. Since speed is positive, the original speed of the train is 24 km per hour and the answer is A.

9. $ABCD$ is a square. From the diagonal BD, a length BX equal to BA is cutoff. From X, a straight line XY is drawn perpendicular to BD to meet AD at Y. Then $AB + AY$ equals
A. $\sqrt{2}BD$ B. $\frac{BD}{\sqrt{2}}$ C. $\sqrt{3}BD$ D. BD
Solution Since $BX = BA$, triangles BXY and BAY



are congruent. Hence $AY = XY$. Also, $\angle XDY = 45°$ and consequently, $XD = XY$. Now,

$$AB + AY = AB + XY = AB + XD = BX + XD = BD$$

Hence answer is D.

10. The number of natural number pairs (x, y) in which $x > y$ and $\frac{5}{x} + \frac{6}{y} = 1$ is

A. 1 B. 2 C. 3 D. 4

Solution We have $5y + 6x = xy$ or

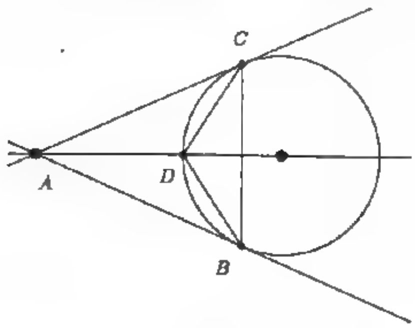
$$(x-5)(y-6) = 30 = 2 \cdot 3 \cdot 5$$

Either $x - 5$ or $y - 6$ must be a multiple of 5. If $x - 5$ is a multiple of 5, then $x = 10, 15, 20$ or 35. The corresponding values of y are respectively, $12, 9, 8, 7$. Of these $x > y$ is satisfied only for $(15, 9), (20, 8), (35, 7)$. If $y - 6$ is a multiple of 5, then $y = 11, 21, 36$. The corresponding values of x are $11, 7, 6$. None of these satisfy the condition $x > y$. Thus the only solutions with $x > y$ are $(15, 9), (20, 8), (35, 7)$ and the answer is C.

11. AB and AC are tangents at B and C to a circle. D is the mid point of the minor arc BC. For the triangle ABC, D is

A. orthocenter B. circumcenter C. incenter D. centroid

Solution $\angle ACD = \angle CBD$, being the angle in the

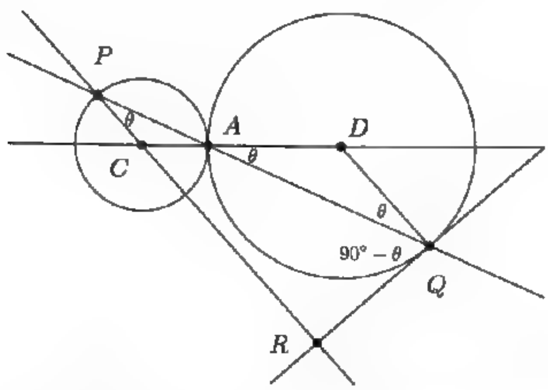


alternate segment and $CD = DB$ since D is the mid point of the arc BC. Thus $\angle BCD = \angle DBC$. Thus CD bisects the angle $\angle ACB$. Similarly, BD bisects the angle $\angle ABC$. Consequently, D is the incenter of the triangle ABC. Answer is C.

12. The radii of two circles are in the ratio $1:2$. C is the center of the smaller circle and D that of the bigger circle. They touch externally at A. PAQ is a straight line with P on the smaller circle and Q on the bigger circle. If the line PAQ does not pass through C, the angle between the tangent at Q to the bigger circle and the diameter through P of the smaller circle (produced if necessary) is

A. 60° B. 75° C. 80° D. None of these

Solution Let the diameter of the smaller circle through



P meet the tangent at Q for the bigger circle at R. Let $\angle APC = \theta$. Then we have

$$\angle PAC = \theta \quad \text{since triangle } PAC \text{ is isosceles}$$
$$\angle AQD = \theta \quad \text{since triangle } AQD \text{ is isosceles}$$
$$\angle AQR = 90^\circ - \theta \quad \text{since } DQ \perp QR$$

Hence $\angle PRQ = 90^\circ$ and the answer is D. Note that ratio of radii is not used in the above and hence the result is true for any two circles touching each other externally.

13. The number of real solutions of the equation

$$\frac{|x-3| - |x+1|}{2|x+1|} = 1$$

is A. 0 B. 1 C. 2 D. 3

The given equation can be written as

$$|x-3|-|x+1|=2|x+1|$$

Thus $x-3=\pm 3(x+1)$. This gives $x=0$ or $x=-3$. Thus the answer is C.

14. The number of real x that satisfy the equation

$$\frac{8^x+27^x}{12^x+18^x}=\frac{7}{6}$$

is A. 2 B. 3 C. 4 D. 0

Solution

$$\begin{aligned}\frac{8^x+27^x}{12^x+18^x} &= \frac{2^{3x}+3^{3x}}{2^{2x}\cdot 3^x+2^x\cdot 3^{2x}}\\ &= \frac{(2^x+3^x)\left(2^{2x}-2^x\cdot 3^x+3^{2x}\right)}{2^x\cdot 3^x\,(3^x+2^x)}\\ &= \frac{2^{2x}-2^x\cdot 3^x+3^{2x}}{2^x\cdot 3^x}\\ &= \left(\frac{2}{3}\right)^x-1+\left(\frac{3}{2}\right)^x\end{aligned}$$

Let $\left(\frac{2}{3}\right)^x=a$. Then we have

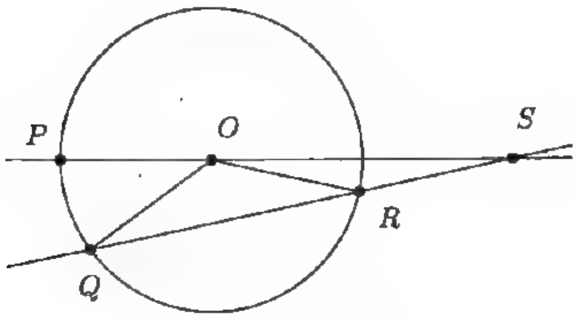
$$a-1+\frac{1}{a}=\frac{7}{6}$$

or $6a^2-13a+6=0$. Thus $a=\frac{2}{3}$ or $\frac{3}{2}$. Hence $x=\pm 1$. Answer is A.

15. P,Q,R are three points on a circle with center O. QR meets the diameter through P at S. If RS equals the radius of the circle and $\angle PSQ=12^\circ$, $\angle POQ$ is

A. 36° B. 42° C. 48° D. 54°

Solution Join OR. Since $OR = RS$, we have



$\angle SOR = \angle OSR = 12^\circ$. Hence $\angle ORQ = 24^\circ$ and consequently,

$$\begin{aligned}\angle POQ &= 180^\circ - \angle QOR - \angle SOR \\ &= 180^\circ - (180^\circ - 2 \times 24^\circ) - 12^\circ \\ &= 36^\circ\end{aligned}$$

Answer is A.

PART - B

16. d is an integer greater than 1. When the numbers 1059, 1417, 2312 are divided by d, they leave the same remainder r. The value of $d - r$ is ———.

Solution Since the numbers leave the same remainder when divided by d, the difference between any two of them will leave a remainder 0 when divided by d. Thus the differences $1417 - 1059 = 358 = 2 \times 179$ and $2312 - 1417 = 895 = 5 \times 179$ are divisible by d. Now d must divide the greatest common divisor of 358 and 895 and hence d divides 179. But since 179 is a prime $d > 1$, it follows that $d = 179$. The remainder when d divides 1059 is 164 and $d - r = 179 - 164 = 15$.

17. The number of integers n that satisfy the inequality $(n^2 - 2)(n^2 - 20) < 0$ is ———.

Solution We must have $2 < n^2 < 20$ and hence

$n^2 = 4, 9, 16$. Hence $n = \pm 2, \pm 3, \pm 4$ and the number of integers is 6.

18. The value of $\sqrt[3]{1+\sqrt{2}} \cdot \sqrt[6]{3-2\sqrt{2}}$ is ———.

Solution

$$\begin{aligned}
\sqrt[3]{1+\sqrt{2}} \cdot \sqrt[6]{3-2\sqrt{2}} &= \sqrt[3]{1+\sqrt{2}} \cdot \sqrt[6]{(1-\sqrt{2})^2} \\
&= \sqrt[3]{1+\sqrt{2}} \cdot \sqrt[3]{1-\sqrt{2}} \\
&= \sqrt[3]{(1+\sqrt{2})(1-\sqrt{2})} \\
&= \sqrt[3]{-1} \\
&= -1
\end{aligned}$$

Also,

$$\begin{aligned}
\sqrt[3]{1+\sqrt{2}} \cdot \sqrt[6]{3-2\sqrt{2}} &- \sqrt[3]{1+\sqrt{2}} : \sqrt[6]{(\sqrt{2}-1)^2} \\
&= \sqrt[3]{1+\sqrt{2}} \cdot \sqrt[3]{\sqrt{2}-1} \\
&= \sqrt[3]{(1+\sqrt{2})(\sqrt{2})-1} \\
&= \sqrt[3]{1} \\
&= 1
\end{aligned}$$

Hence the value is ± 1. Two values arise since $\sqrt[6]{3-2\sqrt{2}}$ has two values.

19. When $a = 5$, $b = 403$, the value of

$$\left\{\frac{1}{a^2} + \frac{1}{b^2} + \frac{2}{a+b}\left(\frac{1}{a} + \frac{1}{b}\right)\right\}\left(\frac{(a+b)^2}{ab}\right)^{-1}$$

is ———.

Solution

$$\left\{\frac{1}{a^2}+\frac{1}{b^2}+\frac{2}{a+b}\left(\frac{1}{a}+\frac{1}{b}\right)\right\}\left(\frac{(a+b)^2}{ab}\right)^{-1}$$
$$=\left\{\frac{1}{a^2}+\frac{1}{b^2}+\frac{2}{ab}\right\}\left(\frac{ab}{(a+b)^2}\right)$$
$$=\left(\frac{1}{a}+\frac{1}{b}\right)^2\left(\frac{ab}{(a+b)^2}\right)$$
$$=\frac{(a+b)^2}{a^2b^2}\left(\frac{ab}{(a+b)^2}\right)$$
$$=\frac{1}{ab}$$

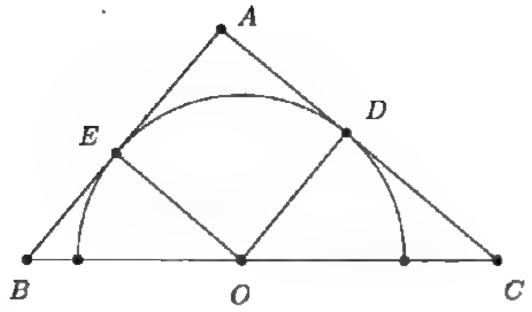
Hence when $a = 5, b = 403$, the value of the given expression is $\dfrac{1}{5\times 403}=\dfrac{1}{2015}$.

20. ABC is an isosceles triangle with base 30 cm. The altitude to the base is 20 cm. The length of the other altitude (in cm) is ———.

Solution The lengths of the equal sides is $\sqrt{20^2+15^2}=25$. Since the area of the triangle is $\frac{1}{2}\times 30\times 20=300$, it follows that length of the other altitude in cm is $\dfrac{300}{\frac{1}{2}\times 25}=24$.

21. In a right angled triangle a semicircle is inscribed so that its diameter lies on the hypotenuse and its center divides the hypotenuse into two segments of lengths 15 cm and 20 cm. The length of the arc of the semicircle between the points at which the legs touch the semicircle is $K\pi$ cm. The value of K is ———.

Solution Let the circle touch the sides at D, E. Since $OEAD$ is a square, it follows that triangles OBE and OCE are similar. If R is the radius of the semicircle, then $\frac{15}{20}=\frac{OB}{OC}=\frac{BE}{R}=\frac{R}{CD}$. Hence $BE=\frac{3}{4}R$. Since $BE^2+R^2=15^2$, it follows that $R=12$. Now $\overset{\frown}{ED}=K\pi=12\times\dfrac{\pi}{2}$ and hence $K=6$.



22. The value of

$$81^{1/\log_5 3} + 27^{\log_9 36} + 3^{4/\log_7 9}$$

is ———.

Solution

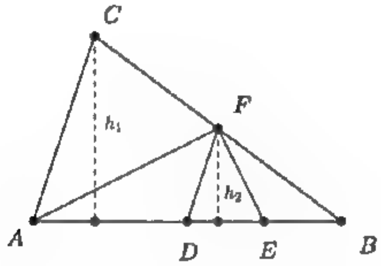
$$81^{1/\log_5 3} = 81^{\log_3 5} = 3^{4\log_3 5} = 3^{\log_3 5^4} = 5^4$$
$$27^{\log_9 36} = 27^{\log_3 6} = 3^{3\log_3 6} = 3^{\log_3 6^3} = 6^3$$
$$3^{4/\log_7 9} = 3^{4\log_9 7} = 3^{\log_9 7^4} = 3^{\log_3 7^2} = 7^2$$

Hence the given expression equals $5^4 + 6^3 + 7^2 = 890$.

23. The sum of all even two digit numbers is ———.
Solution We have

$$\begin{aligned} S &= 10 + 12 + \cdots + 96 + 98 \\ &= 98 + 96 + \cdots + 12 + 10 \\ &= \frac{1}{2}(\underbrace{108 + 108 + \cdots + 108}_{45\, terms}) \\ &= 54 \times 45 \\ &= 2430 \end{aligned}$$

24. ABC is a triangle. D is the midpoint of AB. E is the midpoint of DB and F is the midpoint of BC. If the area of the triangle ABC is 64 square cms, the area of triangle AEF in square cms is ———.
Solution Let h_1 and h_2 be the altitudes of the triangles



ABC and AEF. Clearly $h_2 = \frac{h_1}{2}$ and

$$AE = AD + DE = \frac{1}{2}AB + \frac{1}{4}AB = \frac{3}{4}AB$$

Since $\frac{1}{2}h_1 \times AB = 64$, it follows that

$$\frac{1}{2}h_2 \times AE = \frac{1}{2} \times \frac{h_1}{2} \times \frac{3}{4}AB = \frac{3}{8} \times 64 = 24$$

square cms.

25. In a geometric progression of real numbers, the sum of the first two terms is 7 and the sum of first six terms is 91. The sum of first four terms is ———.

Solution Let the geometric progression be $a, ar, ar^2, \ldots$. Given

$$a + ar = 7, \quad a + ar + ar^2 + ar^3 + ar^4 + ar^5 = 91$$

Now,

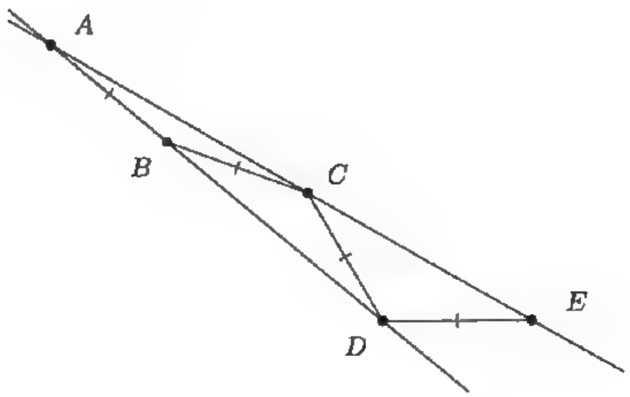
$$\begin{aligned} \frac{a(1 + r + \cdots + r^5)}{a(1+r)} &= \frac{1 - r^6}{(1-r)(1+r)} \\ &= \frac{1 - r^6}{1 - r^2} \\ &= 1 + r^2 + r^4 \end{aligned}$$

Hence $r^4 + r^2 = \dfrac{91}{7} - 1 = 12$ and $(r^2 + 4)(r^2 - 3) = 0$. Since r is real, we have $r^2 = 3$. Now, sum of first four terms is

$$a(1 + r + r^2 + r^3) = a(1 + r + 3 + 3r) = 4a(1+r) = 28$$

26. In a triangle ADE, $\angle ADE = 140°$. B and C are points on AD and AE respectively. A, B, C, D, E are all distinct. If $AB = BC = CD = DE$, $\angle EAD$ equals ———.

Let $\angle EAD = x$. We have



$$\angle CED = 180° - 140° - x = 40° - x$$
$$\angle DCE = \angle CED = 40° - x$$
$$\angle BCA = x$$
$$\angle CBD = \angle CDB = 2x$$

Thus at C, we have

$$\angle ACB + \angle BCD + \angle DCE = x + (180° - 4x) + (40° - x) = 180°$$

and $4x = 40°$. Thus $x = 10°$.

27. When 265 is divided by a two digit number, the remainder is 5. The number of such two digit numbers is ———.
Solution We need to find the two digit numbers that divide $265 - 5 = 260$. Since $260 = 2 \times 2 \times 5 \times 13$, the two digit factors are $10, 13, 20, 26, 52, 65$. Hence there are 6 such numbers.

28. Viswa is playing with three identical rectangular boxes. The dimensions of each are $50\,cm \times 5\,cm \times 2\,cm$. He glues

them in several ways to obtain rectangular boxes. After gluing the maximum surface area he can get is ———.

Solution To obtain the maximum surface area, Viswa needs to glue the smallest surfaces together. Once this is done the resulting box has dimensions $150\,cm \times 5\,cm \times 2\,cm$. Hence the maximum surface area is

$$2 \times (150 \times 5 + 150 \times 2 + 5 \times 2) = 2120$$

square cms.

29. A five digit number is reversed. The difference between the original number and the reversed number is not zero. The largest prime number that divides this difference is ———.

Solution If the digits of the number are a, b, c, d, e, then the number is $10^4a+10^3b+102c+10d+e$ and the reversed number is $10^4e+10^3d+10^2c+10b+a$. Their difference is $9999(a-e)+990(b-d)$. Considering the case $a=e$ and $b=d+1$, the difference is 990 and the largest prime factor of this difference is 11.

30. If $x^3+3xy^2 = 14$, $y^3+3yx^2 = 13$, x, y are real, the value of x^2+y^2 is ———.

Solution We have

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3 = 13+14 = 27$$

Since $x+y$ is real, it follows that $x+y=3$. Again

$$(x-y)^3 = x^3-3x^2y+3xy^2-y^3 = 14-13 = 1$$

and $x-y=1$. Thus $x=2, y=1$ and $x^2+y^2=5$.

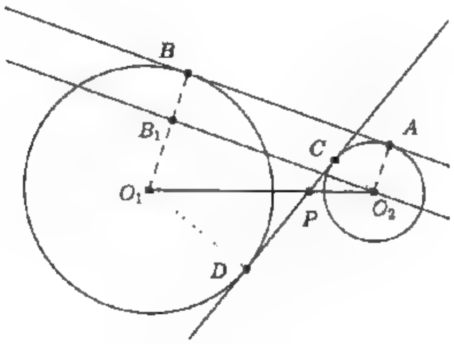
SCREENING TEST – RAMANUJAN CONTEST
NMTC at INTER LEVEL
XI & XII Standards

PART – A

1. The radii of two circles are 5 cms and 2 cms. The length of the direct common tangents to the circles is 1.5 times the length of the transverse common tangents. The distance between the centers of the circles in cms is

   A. 8 B. 9 C. 10 D. 11

   Solution Let $CD = x$ be the transverse common



tangent and $O_1O_2 = d$, the distance between the centers. Let the transverse common tangent intersect O_1O_2 at P. Since the triangles O_1PD and O_2PC are similar, we have $\dfrac{PC}{PD} = \dfrac{2}{5}$. Thus $PC = \dfrac{2}{7}x$, $PD = \dfrac{5}{7}x$. Also, in the same way, $O_1P = \dfrac{5}{7}d$, $PO_2 = \dfrac{2}{7}d$. We have

$$\frac{25}{49}d^2 = O_1P^2 = PD^2 + O_1D^2 = \frac{25}{49}x^2 + 25 \qquad (1)$$

and

$$\frac{9}{4}x^2 = AB^2 = O_2B_1^2 = d^2 - 9$$

Hence from (1),

$$\frac{25}{49}d^2 = \frac{25}{49}\left(\frac{4}{9}(d^2 - 9)\right) + 25$$

and $d^2 = 81$ and $d = 9$. Answer B.

2. The geometric mean of two positive numbers is greater by 12 than the smaller number and their arithmetic mean is smaller by 24 than the larger number. The sum of the two numbers is

   A. 56 B. 29 C. 72 D. 60

   Solution Let the numbers be x, y with $x < y$. We have $\sqrt{xy} = x + 12$ and $\dfrac{x+y}{2} = y - 24$. Hence

$$xy = (x+12)^2, \quad y = x + 48$$

   Thus

$$x(x+48) = x^2 + 24x + 144$$

   and $x = 6$. It follows that $y = x + 48 = 54$ and the sum of the numbers is 60. Answer D.

3. The number of real solutions of the equation

$$\left(16x^{200} + 1\right)\left(y^{200} + 1\right) = 16(xy)^{100}$$

   is A. 100 B. 200 C. 400 D. 4

   Solution Putting $4x^{100} = u$, $y^{100} = v$, we can write the given equation as

$$(u^2 + 1)(v^2 + 1) = 4uv$$

   or

$$u^2v^2 + u^2 + v^2 - 4uv + 1 = 0$$

   This can be written as

$$(uv - 1)^2 + (u - v)^2 = 0$$

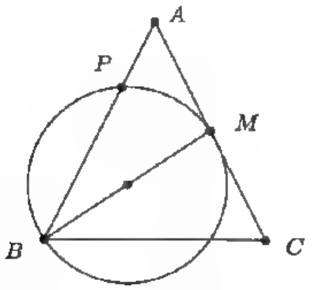
   Since u, v are real, it follows that $u = v$ and $uv = 1$. Thus $v^2 = 1$ or $y^{200} = 1$ and $y = \pm 1$. Also,

$u^2 = 16x^{200} = 1$ gives $x = \pm\frac{1}{\sqrt[50]{2}}$. Thus the solutions are

$$\left(\pm\frac{1}{\sqrt[50]{2}}, \pm 1\right)$$

and the answer is D.

4. ABC is an isosceles triangle in which $AB = AC$. A circle is drawn through B and touching AC at its midpoint M. The circle cuts AB at P. The ratio $\frac{BP}{AP}$ is
A. 1 B. 2 C. 3 D. 4
Solution For the circle passing through B and touching



AC at M, APB is a secant and AM is a tangent. Hence

$$AP \times AB = AM^2 = \left(\frac{AB}{2}\right)^2$$

Thus $AP = \frac{AB}{4}$ and hence $BP = \frac{3}{4}AB$. Hence $\frac{BP}{AP} = 3$ and answer is C.

5. Integers a, b, c, d satisfy $|ac + bd| = |ad + bc| = 1$. Then

A. $a = b = c = d$ is the only possibility

B. $|a| = |b|$ and $|c| = |d|$ is the only possibility

C. $|a| = |b| = 1$ and $|c| = |d| = 1$

D. $|a| = |b| = 1$ or $|c| = |d| = 1$

Solution If $ac+bd$ and $ad+bc$ have the same sign, then $ac + bd = ad + bc$ and hence $(a - b)(c - d) = 0$. Thus either $a = b$ or $c = d$.

If $ac + bd$ and $ad + bc$ are of opposite signs, then $ac + bd + ad + bc = 0$ and hence $(a + b)(c + d) = 0$. In this case either $a = -b$ or $c = -d$.

Consequently, we have either $|a| = |b|$ or $|c| = |d|$. If $|a| = |b|$, then $|a|$ divides $|ac + bd| = 1$ and hence $|a| = |b| = 1$. If $|c| = |d|$, then $|c|$ divides $|ad + bc|$ and in this case $|c| = |d| = 1$. Thus the answer is C.

6. Two bicyclists Samrud and Saket ride around a circular track of perimeter 18 kms in opposite directions. Samrud rides at 10 kms an hour and Saket at 14 kms an hour. The time after the start when they meet for the second time is

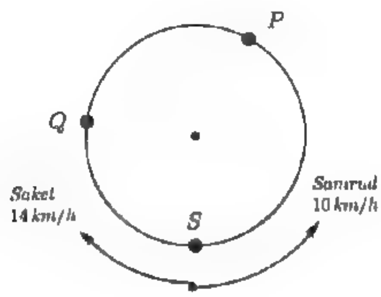
   A. 1 hour 15 minutes

   B. 1 hour 30 minutes

   C. 1 hour 45 minutes

   D. 2 hours

Solution Let them meet at P for the first time. If



$x = \widehat{SP}$, then we have $\dfrac{x}{10} = \dfrac{18 - x}{14}$. Thus $x = \dfrac{15}{2}$ and

their first meeting happens after $\frac{15/2}{10} = \frac{3}{4}$ hours.
If they meet at Q for their second meeting, then the time taken by Samrud to travel from P to Q is same as time taken by Saket to travel from Q to P. If $y = \overset{\frown}{PQ}$, then we have $\frac{y}{10} = \frac{18-y}{14}$. Thus $y = \frac{15}{2}$.

Again the time taken by Samrud to travel from P to Q is $\frac{15/2}{10} = \frac{3}{4}$ hours. Thus their second meeting happens after $\frac{3}{4} + \frac{3}{4} = 1\frac{1}{2}$ hours. Answer is B.

7. The angles of a triangle are in the ratio $2:3:7$. The length of the smallest side is 2015 cms. The radius of the circumcircle of the triangle in cms is
A. 2015 B. 4030 C. $\frac{2015}{2}$ D. 8045
Solution The angles of the triangle are $30°, 45°, 105°$. The smallest side is opposite to the smallest angle and since $a = 2R\sin A$ where R is the circumradius, we have $2015 = 2R\sin 30° = R$. Answer A.

8. The area under the curve

$$y = \frac{|x-3| + |x+1|}{|x+3| + |x-1|}$$

X axis and the ordinates $x = -3$ and $x = 1$ is
A. 5 B. 6 C. 4 D. 7
Solution Between $x = -3$ and $x = 1$, the given curve is

$$y = \begin{cases} -\frac{x}{2} + \frac{1}{2}, & \text{if } -3 \le x \le -1 \\ 1, & \text{if } -1 \le x \le 1 \end{cases}$$

The graph is shown in Figure for Q 8. The required area is the sum of the areas of $ABCD$ and $BEFC$. Area of $ABCD$ is 3 sq units and that of $BCEF$ is 2 sq units. Thus the required area is 5 sq units. Answer is A.

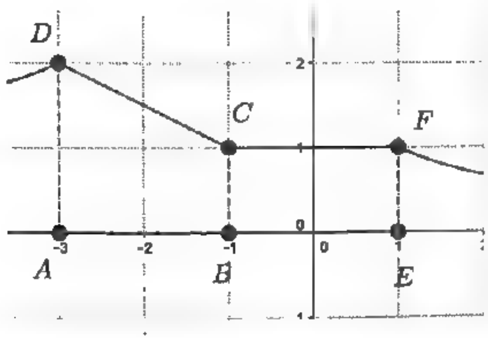


Figure for Q 8

9. The number 27000001 has exactly four prime factors. The sum of these factors is

   A. 573 B. 612 C. 643 D. 652

   Solution

$$\begin{aligned} 27000001 &= 300^3 + 1 \\ &= (300+1)(300^2 - 300 + 1) \\ &= 301 \times ((300+1)^2 - 900) \\ &= 301 \times (300 + 1 + 30) \times (300 + 1 - 30) \\ &= 301 \times 331 \times 271 \\ &= 7 \times 43 \times 331 \times 271 \end{aligned}$$

   Thus the prime factors are $7, 43, 271$ and 331 and their sum is 652. Answer D.

10. The common difference of the arithmetic progression $\{a_n\}$ is d. The common ratio of the geometric progression $\{b_n\}$ is r, where r is a positive rational number less than 1. Given $a_1 = d, b_1 = d^2$ and $\dfrac{a_1^2 + a_2^2 + a_3^2}{b_1 + b_2 + b_3}$ is a positive integer, the value of r is

    A. $\frac{1}{3}$ B. $\frac{1}{2}$ C. $\frac{1}{4}$ D. $\frac{1}{6}$

Solution We have $a_1 = d, a_2 = 2d, a_3 = 3d$ and $b_1 = d^2, b_2 = d^2r, b_3 = d^2r^2$. Hence for some positive integer m, we have

$$m = \frac{a_1^2 + a_2^2 + a_3^2}{b_1 + b_2 + b_3} = \frac{d^2 + 4d^2 + 9d^2}{d^2 + d^2r + d^2r^2} = \frac{14}{1 + r + r^2}$$

Thus $r^2 + r + \left(1 - \frac{14}{m}\right) = 0$ and

$$r = -\frac{1}{2} + \sqrt{\frac{56 - 3m}{4m}}$$

From $0 < r < 1$, we have

$$\frac{1}{2} < r + \frac{1}{2} = \sqrt{\frac{56 - 3m}{4m}} < \frac{3}{2}$$

and hence

$$\frac{1}{4} < \frac{56 - 3m}{4m} < \frac{9}{4} \Rightarrow 4m < 56 \text{ and } 12m > 56$$

Thus $5 \le m \le 13$. It is easy to see that only for $m = 8$, $\sqrt{\frac{56-3m}{4m}}$ is rational and $r = \frac{1}{2}$. Answer B.

11. The value of

$$\frac{\sqrt{3}}{\sqrt{\sqrt{3}+1} - 1} - \frac{\sqrt{3}}{\sqrt{\sqrt{3}+1} + 1}$$

is A. $2\sqrt{3}$ B. $\sqrt{3}$ C. 2 D. $\frac{\sqrt{3}}{2}$

Solution Let $\sqrt{3} + 1 = x$.

$$\begin{aligned}\frac{\sqrt{3}}{\sqrt{\sqrt{3}+1} - 1} - \frac{\sqrt{3}}{\sqrt{\sqrt{3}+1} + 1} &= \frac{x-1}{\sqrt{x} - 1} - \frac{x-1}{\sqrt{x} + 1} \\ &= \sqrt{x} + 1 - (\sqrt{x} - 1) \\ &= 2\end{aligned}$$

Answer C.

12. In a right angled triangle triangle, the ratio of the circumradius to the inradius is $5:2$. One acute angle of the triangle is

A. $\tan^{-1}\left(\frac{3}{4}\right)$ B. $\tan^{-1}\left(\frac{3}{5}\right)$ C. $\tan^{-1}\left(\frac{4}{5}\right)$

D. $\tan^{-1}\left(\frac{1}{2}\right)$

Solution If the incircle touches the sides at D, E, F, then from Figure for Q 12, we have $y+z=a$, $x+y=c$, $x+z=b$. Hence $z=r=\frac{a+b-c}{2}$. Also, since the triangle is right angled, $R=\dfrac{c}{2}$. Given $\dfrac{a+b-c}{c}=\dfrac{2}{5}$.

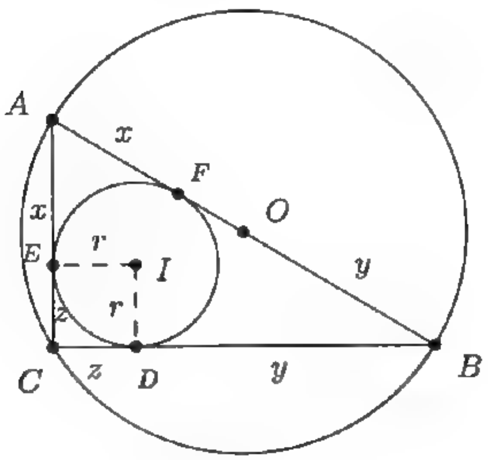


Figure for Q 12

If $\angle B=\theta$, then we have

$$\frac{a+b-c}{c}=\frac{2}{5} \Rightarrow \frac{a}{c}+\frac{b}{c}=\frac{7}{5}$$
$$\Rightarrow \cos\theta+\sin\theta=\frac{7}{5}$$
$$\Rightarrow \sin 2\theta=\frac{24}{25}$$

If $t=\tan\theta$, then

$$\sin 2\theta=\frac{2t}{1+t^2}=\frac{24}{25} \Rightarrow t=\frac{3}{4} \text{ or } \frac{4}{3}$$

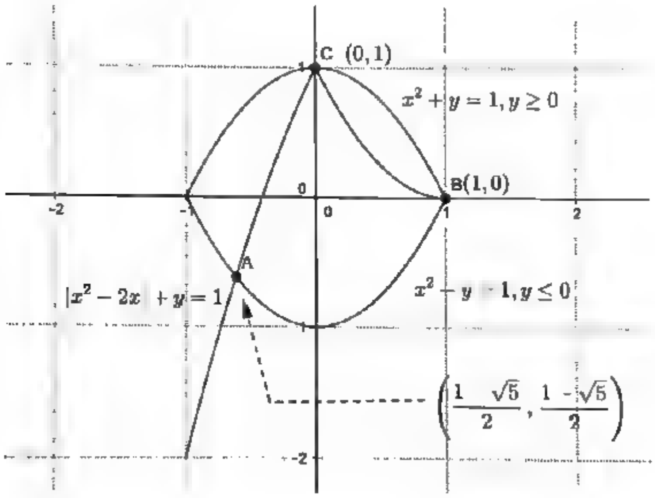
If $t = \frac{3}{4}$, then $\angle B = \tan^{-1}\frac{3}{4}$. If $t = \frac{4}{3}$, then $\tan A = \tan(90° - B) = \cot\theta = \frac{3}{4}$. Thus one of the angles is $\tan^{-1}\left(\frac{3}{4}\right)$.

13. The number of real solutions of the equations

$$|x^2 - 2x| + y = 1, \qquad x^2 + |y| = 1$$

is A. 1 B. 2 C. 3 D. 4

Solution We will consider the four cases separately:



(a) $x^2 - 2x \geq 0$, $y \geq 0$: Here, we have to solve

$$x^2 - 2x + y = 1, \quad x^2 + y = 1 \quad .$$

The unique solution is $x = 0, y = 1$

(b) $x^2 - 2x \leq 0$, $y \geq 0$: Here, we have to solve

$$-x^2 + 2x + y = 1, \quad x^2 + y = 1$$

We have $-2x^2 + 2x = 0$ and $(x, y) = (0, 1)$, or $(1, 0)$. The new solution is $(1, 0)$.

(c) $x^2 - 2x \geq 0$, $y \leq 0$: Here, we have to solve

$$x^2 - 2x + y = 1, \quad x^2 - y = 1$$

Here $x^2 - x - 1 = 0$, $x = y$. Solving the quadratic in x we get $x = \frac{1 \pm \sqrt{5}}{2}$. $y \leq 0$, we have the solution

$$(x, y) = \left(\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$$

(d) $x^2 - 2x \leq 0$, $y \leq 0$: Here, we have to solve

$$-x^2 + 2x - y = 1, \quad x^2 - y = 1$$

Here again the unique solution is $(0, 1)$. Thus there are three solutions and answer is C.

14. In an isosceles triangle, the altitude drawn to the base is $\dfrac{2}{3}$ times the radius of the circumcircle. The base angle of the triangle is

A. $\cos^{-1}\left(\frac{2}{3}\right)$ B. $\cos^{-1}\left(\frac{2}{\sqrt{3}}\right)$ C. $2\cos^{-1}\sqrt{\frac{2}{3}}$

D. $\cos^{-1}\sqrt{\frac{2}{3}}$

Solution If the base angle is θ, from Figure for Q 14,

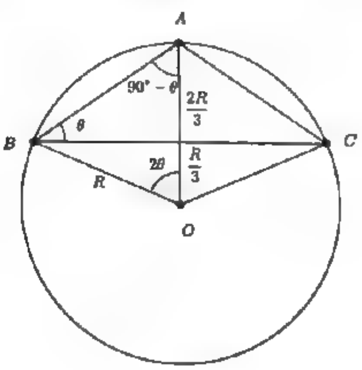


Figure for Q 14

we have $\cos 2\theta = \dfrac{R/3}{R} = \dfrac{1}{3}$. Hence $2\cos^2\theta - 1 = \dfrac{1}{3}$ and $\cos^2\theta = \dfrac{2}{3}$. Thus $\theta = \cos^{-1}\sqrt{\dfrac{2}{3}}$. Answer D.

15. For real values of x, y, the minimum value of the expression

$$x^2 + 2xy + 3y^2 + 2x + 6y + 4$$

is A. 0 B. $\dfrac{1}{2}$ C. $\dfrac{3}{4}$ D. 1

Solution We have

$$\begin{aligned}
&x^2 + 2xy + 3y^2 + 2x + 6y + 4 \\
&\quad = (x^2 + 2xy + y^2 + 2x + 2y + 1) + (2y^2 + 4y + 2) + 1 \\
&\quad = (x + y + 1)^2 + 2(y + 1)^2 + 1 \\
&\quad \geq 1
\end{aligned}$$

Thus the least value is 1 and answer is D.

PART – B

16. In an infinite geometric progression with common ratio $r < 1$, every term is 4 times as large as the sum of all its successive terms. The value of r is ———.

Solution Let the geometric progression be

$$a, ar, ar^2, \ldots, ar^{n-1}, \ldots$$

Given that

$$\begin{aligned}
ar^{n-1} &= 4(ar^n + ar^{n+1} + \cdots) \\
&= 4ar^n(1 + r + r^2 + \cdots) \\
&= \frac{4ar^n}{1 - r}
\end{aligned}$$

Hence $\dfrac{4r}{1 - r} = 1$ and hence $r = \dfrac{1}{5}$.

17. The length of the hypotenuse of a right angled isosceles triangle is 2015. The perimeter of the incircle is $\dfrac{2015\pi}{K}$. The value of K is ———.

Solution In a right angled triangle with sides a, b and hypotenuse c, the inradius is given by $\frac{a+b-c}{2}$. Since the perimeter of the incircle is $\dfrac{2015\pi}{K}$, its radius is $\dfrac{2015}{2K}$. If a, a, c are the sides of the triangle, we have

$$\frac{2a-c}{2} = \frac{2015}{2K} \Rightarrow K = \frac{2015}{2a-c}$$

Since $c = 2015$, we have $a = \dfrac{2015}{\sqrt{2}}$ and

$$K = \frac{2015}{2a-c} = \frac{2015}{2015\sqrt{2}-2015} = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1$$

18. If

$$n = \frac{(\log_7 4)(\log_7 5 - \log_7 2)}{(\log_7 25)(\log_7 8 - \log_7 4)}$$

the value of 5^n is ———.

Solution

$$\begin{aligned} n &= \frac{(\log_7 4)(\log_7 5 - \log_7 2)}{(\log_7 25)(\log_7 8 \quad \log_7 4)} \\ &= \frac{(2\log_7 2)\left(\log_7 \frac{5}{2}\right)}{(2\log_7 5)(\log_7 2)} \\ &= \frac{\log_7 \frac{5}{2}}{\log_7 5} \\ &= \log_5 \frac{5}{2} \quad \text{by base change rule } \frac{\log_a b}{\log_a c} = \log_c b \end{aligned}$$

Hence $5^n = 5^{\log_5 \frac{5}{2}} = \dfrac{5}{2}$

19. If $\sec x + \tan x = \dfrac{22}{7}$ and $\operatorname{cosec} x + \cot x = \dfrac{m}{n}$, where m, n are integers and have no common factors > 1, the

value of $m+n$ is ———.

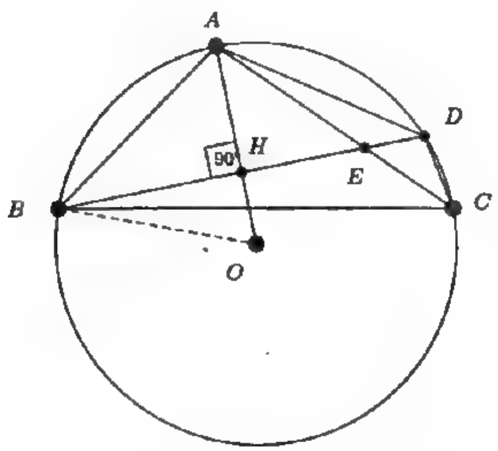
Solution

$$\begin{aligned}\sec x+\tan x=\frac{22}{7} &\Rightarrow \frac{1+\sin x}{\cos x}=\frac{22}{7}\\ &\Rightarrow \frac{1+\sin x+\cos x}{1+\sin x-\cos x}=\frac{22+7}{22-7}=\frac{29}{15}\\ &\Rightarrow \cot\frac{x}{2}=\frac{29}{15}\\ &\Rightarrow \frac{1+\cos x}{\sin x}=\frac{29}{15}\\ &\Rightarrow \operatorname{cosec} x+\cot x=\frac{29}{15}\end{aligned}$$

Hence $m+n=44$.

20. The circumradius of a cyclic quadrilateral $ABCD$ is 2. AC, BD cut at E such that $AE = EC$. If $AB=\sqrt{2}AE$, $BD=2\sqrt{3}$, the area of the quadrilateral is ———.

Solution We have $AE=EC$ and $AB=\sqrt{2}AE$. Hence



$AB^2=2AE^2=AE\cdot AC$. Thus $AB/AC=AE/AB$ and $\angle EAB=\angle BAC$. Thus $\triangle ABE\sim\triangle ACB$. Hence

$\angle ABE = \angle ACB = \angle ADB$ and it follows that $\triangle BAD$ is isosceles with $AB = AD$. The vertex A bisects the arc $\overset{\frown}{BD}$ and consequently, if O is the center of the circumcircle of $ABCD$, $OA \perp BD$. Let H be the point of intersection of OA and BD. H bisects BD and hence $BH = \sqrt{3}$ and

$$OH = \sqrt{OH^2 - BH^2} = \sqrt{2^2 - 3} = 1$$

$AH - OA - OH = 2 - 1 = 1$ and area of $\triangle ABD = \frac{1}{2} \times BD \times AH = \sqrt{3}$.
Also, since $AE = EC$, triangles BCE and ABE have the same area and triangles CDE and ADE have the same area. Thus triangle CBD has the same area as triangle ABD. Hence area of quadrilateral $ABCD = 2 \times \triangle ABD = 2\sqrt{3}$.

21. The smallest natural number n that satisfies $12^{200} < n^{300}$ is ———.

Solution

$$12^{200} < n^{300} \Rightarrow 144^{100} < (n^3)^{100} \Rightarrow 144 < n^3$$

Since $5^3 < 144 < 6^3$, the smallest number is 6.

22. If

$$\sqrt{x+1} + \sqrt{y} + \sqrt{z-4} = \frac{x+y+z}{2}$$

the value of $x + y + z$ is ———.

Solution We have

$$\begin{aligned} 2\sqrt{x+1} + 2\sqrt{y} + 2\sqrt{z-4} &= x + y + z \\ &= (x+1) + (y+3) + (z-4) \end{aligned}$$

Thus

$$\begin{aligned} &((x+1) - 2\sqrt{x+1} + 1) + (y - 2\sqrt{y} + 1) \\ &\qquad + (z - 4 - 2\sqrt{z-4} + 1) = 0 \end{aligned}$$

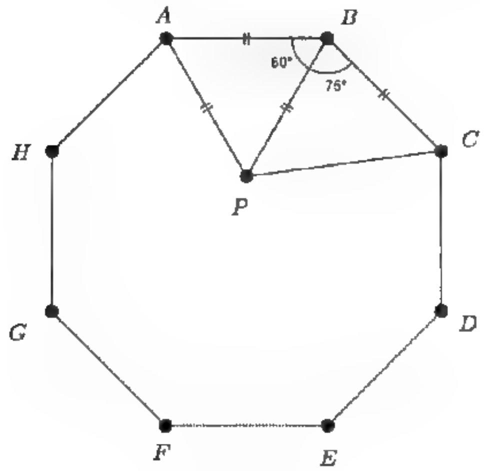
Hence

$$(\sqrt{x+1}-1)^2+(\sqrt{y}-1)^2+(\sqrt{z-4}-1)^2=0$$
$$\Rightarrow x=0,\ y=1,\ z=5$$

Hence $x+y+z=6$.

23. $ABCDEFGH$ is a regular octagon. P is a point inside the octagon such that the triangle ABP is equilateral. The measure of the angle APC is ——.

Solution We have $\angle ABC = 135^\circ$, being the interior



angle of a regular octagon.

Since $PA = AB = PB = BC$, $\angle BPC = \angle BCP$. Thus $\angle BPC = \frac{1}{2}(180^\circ - 75^\circ) = 52.5^\circ$. Thus $\angle APC = 60^\circ + 52.5^\circ = 112.5^\circ$.

24. The sum of the infinite series

$$\frac{1}{2}+\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{5}{32}+\frac{8}{64}+\frac{13}{128}+\frac{21}{256}+\frac{34}{512}+\cdots$$

is ————.

Solution

The denominators of the terms are $2^1, 2^2, 2^3, \ldots$. The

numerators are $1, 2, 3, 5, 8, 13, 21, 34, \ldots$. From the third term each term is the sum of the previous two terms. If we write the nth term as $\frac{a_n}{2^n}$, then $a_n = a_{n-1} + a_{n-2}$, for $n \geq 3$. Thus we have

$$\begin{aligned}
S = \sum_{n=1}^{\infty} \frac{a_n}{2^n} &= \frac{1}{2} + \frac{1}{4} + \sum_{n=3}^{\infty} \frac{a_{n-1} + a_{n-2}}{2^n} \\
&= \frac{1}{2} + \frac{1}{4} + \sum_{n=3}^{\infty} \frac{a_{n-1}}{2^n} + \sum_{n=3}^{\infty} \frac{a_{n-2}}{2^n} \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{2} \sum_{n=3}^{\infty} \frac{a_{n-1}}{2^{n-1}} + \frac{1}{2^2} \sum_{n=3}^{\infty} \frac{a_{n-2}}{2^{n-2}} \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{2} \sum_{n=2}^{\infty} \frac{a_n}{2^n} + \frac{1}{2^2} \sum_{n=1}^{\infty} \frac{a_n}{2^n} \\
&= \frac{3}{4} + \frac{1}{2}\left(S - \frac{1}{2}\right) + \frac{1}{4} S \\
&= \frac{1}{2} + \frac{3}{4} S
\end{aligned}$$

Hence $\frac{S}{4} = \frac{1}{2}$ and $S = 2$.

25. A seven digit number consists of $0, 1, 2, 3, 4, 6, 8$ each used once. The largest such number that is a multiple of 120 is ———.

Solution The largest seven digit number using the given digits once is 8643210. Since $\frac{8643210}{120} = 72026.75$, it follows that the required number is no greater than $72026 \times 120 = 8643120$. Since this uses all the given digits once, this is the largest number with the stated property.

26. a, b, c, d, e are real numbers such that

$$a + 4b + 9c + 16d + 25e = 1 \qquad (1)$$

$$4a + 9b + 16c + 25d + 36e = 8 \qquad (2)$$

$$9a + 16b + 25c + 36d + 49e = 23 \qquad (3)$$

The value of $a + b + c + d + e$ is ———.

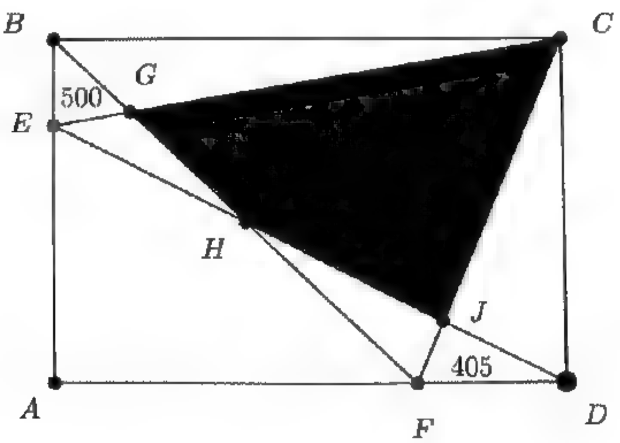
Solution Adding (1) and (3) and subtracting twice (2), we obtain

$$2a + 2b + 2c + 2d + 2e = (1 + 23) - 2 \times 8 = 8$$

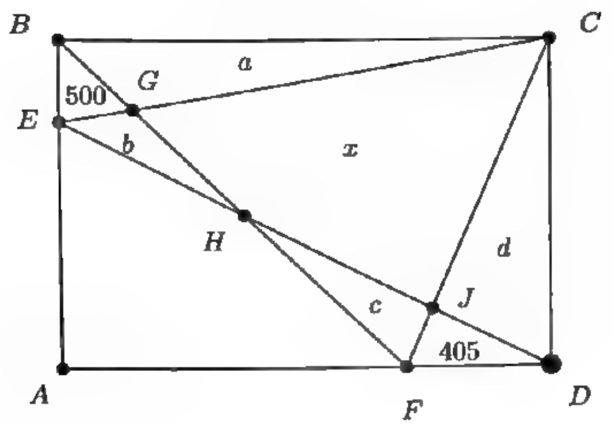
Hence $a + b + c + d + e = 4$.

27. In the adjoining figure, $ABCD$ is a rectangle. Area of $\triangle BEG = 500$, area of $\triangle JFD = 405$ and area of quadrilateral $EHFA = 1110$. Area of the shaded region is ———.

Solution Let the area of the triangles BGC, EGH, HJF



and CJD be respectively a, b, c and d. Let the area of the shaded region be x. Let S be the area of the



rectangle $ABCD$. We have

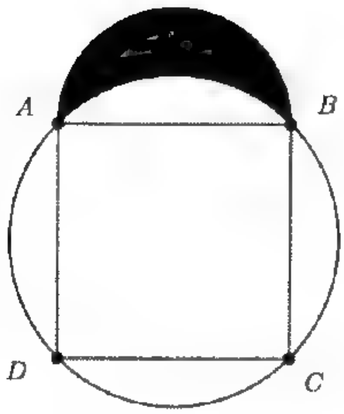
$$a + x + c = \frac{S}{2}$$
$$500 + b + 1110 + d + 405 = \frac{S}{2}$$
$$b + x + d = \frac{S}{2}$$
$$500 + a + 1110 + c + 405 = \frac{S}{2}$$

Hence

$$a + x + c = 500 + a + 1110 + c + 405 \Rightarrow x = 2015$$

28. $ABCD$ is a square inscribed in a circle of unit radius. On AB as diameter, a semicircle is drawn. Area of the shaded region is ———.

Solution The arc $\overset{\frown}{AB}$ of the circle circumscribing the square subtends an angle $\frac{\pi}{2}$ at the center of the circle. Thus the area of the sector subtended by $\overset{\frown}{AB}$ is $\frac{1}{2} \times 1^2 \times \frac{\pi}{2} = \frac{\pi}{4}$. The triangle subtended by the chord AB at the center of the circle has area $\frac{1}{2}$. Hence the



area between the arc $\widehat{AB}$ and the chord AB is $\frac{\pi}{4} - \frac{1}{2}$. The chord AB has length $\sqrt{2}$ and hence the semi circle on AB as diameter has area $\frac{1}{2}\pi \times (\sqrt{2}/2)^2 = \frac{\pi}{4}$. Thus the area of the shaded portion is $\frac{\pi}{4} - \left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{1}{2}$

29. The smallest number that can be written as the sum of two different primes in two different ways is ———.

Solution The first few primes are $2, 3, 5, 7, 11, 13, 17, \ldots$. Taking sums of two primes we obtain the sequences

$$5, 7, 13, 15, 19, \ldots$$

$$8, 10, 14, 16, 20, \ldots$$

$$12, 16, 18, 22, \ldots$$

In the above 16 is the smallest number that occurs in more than one sequence. We have $16 = 3 + 13 = 5 + 11$. Thus 16 is the required number.

30. The number less than 100 that increases by 20% when its digits are reversed is ———.

Solution Let the number be $10a + b$. We have

$$10b + a = 1.2(10a + b) \Rightarrow 4b = 5a \Rightarrow a = 4,\ b = 5$$

Thus the required number is 45.

FINAL – GAUSS CONTEST
NMTC at PRIMARY LEVEL
V & VI Standards

1. A unit fraction is one of the form $\frac{1}{a}$ where $a \neq \pm 1$ is a natural number. Any proper fraction can be written as the sum of two or more unit fractions. For example,

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{6}, \quad \frac{5}{6} = \frac{1}{2} + \frac{1}{3}, \quad \frac{1}{24} = \frac{1}{54} + \frac{1}{72} + \frac{1}{108}$$

Express $\frac{1}{15}$ as the sum of two different unit fractions in 4 different ways.

Solution

$$\frac{1}{15} = \frac{3+5}{15 \times (3+5)} = \frac{1}{5 \times 8} + \frac{1}{3 \times 8} = \frac{1}{40} + \frac{1}{24}$$
$$\frac{1}{15} = \frac{5+15}{15 \times (5+15)} = \frac{1}{3 \times 20} + \frac{1}{1 \times 20} = \frac{1}{60} + \frac{1}{20}$$
$$\frac{1}{15} = \frac{3+15}{15 \times (3+15)} = \frac{1}{5 \times 18} + \frac{1}{1 \times 18} = \frac{1}{90} + \frac{1}{18}$$
$$\frac{1}{15} = \frac{1+15}{15 \times (1+15)} = \frac{1}{15 \times 16} + \frac{1}{1 \times 16} = \frac{1}{240} + \frac{1}{16}$$

2. (a) The numbers 11284 and 7655 when divided by a 3 digit number leave the same remainder. Find the number and the remainder.

(b) What is the least positive integer to be subtracted from 1936 so that the resulting number when divided by 9, 10 and 15 will leave the same remainder in each case?

Solution

(a) Since 11284 and 7655 leave the same remainder when divided by the three digit number, N, the number

N must divide their difference $11284-7655 = 3629$. Now $3629 = 19 \times 191$ and both 19 and 191 are primes. Thus the required number is 191.

(b) Suppose that k is the least positive integer to be subtracted from 1936 so that $1936 - k$ leaves the same remainder r when divided by 9, 10 and 15. Note that $0 \leq r \leq 8$. Since $1936 - k - r$ is divisible by 9, 10 and 15, it is divisible by their least common multiple 90. The largest multiple of 90 that is ≤ 1936 is 1890. Hence $1936 - k - r = 1890$ and $k = 46 - r$. k is least when r is maximum. The maximum remainder possible is 8 and hence $k = 46 - 8 = 38$.
Clearly, $1936-38 = 1898$ leaves the same remainder 8, when divided by 9, 10 and 15.

3. A, M, T, I represent different non zero digits. It is given that

$$A + M + T + I = 11 \tag{1}$$
$$A + M + I = 10 \tag{2}$$
$$A + M = I \tag{3}$$

Further A, M, T, I also satisfy the following addition:

$$\begin{array}{cccccccc}
A & M & T & I & A & M & T & I \\
 & A & M & T & I & A & M & T \\
 & & A & M & T & I & A & M \\
 & & & A & M & T & I & A \\
 & & & & A & M & T & I \\
 & & & & & A & M & T \\
 & & & & & & A & M \\
 & & & & & & & A \\
\hline
* & * & * & * & 5 & * & * & * \\
\hline
\end{array}$$

Find the digits in the places represented by stars.

Solution From (1) and (2), it follows that $T = 1$. From (2) and (3), we have $2I = 10$ or $I = 5$. Hence $A + M = 5$. Thus $(A, M) = (1,4), (4,1), (2,3), (3,2)$, But since $T = 1$, and A, M, T, I represent different digits, it follows that $(A, M) = (2,3)$ or $(3,2)$.

If $A = 2, M = 3, T = 1, I = 5$, then the second addition becomes

$$\begin{array}{r}
23152315 \\
2315231 \\
231523 \\
23152 \\
2315 \\
231 \\
23 \\
1 \\
\hline
25724792 \\
\hline
\end{array}$$

This does not have 5 in the thousandth's place. Hence $(A, M) = (3, 2)$. In this case we have

$$\begin{array}{r}
32153215 \\
3215321 \\
321532 \\
32153 \\
3215 \\
321 \\
32 \\
3 \\
\hline
35725792 \\
\hline
\end{array}$$

which has 5 in the thousandth's place. Thus the final sum is 35725792.

4. ABC is a three digit number in which the digit A is greater than the digits B and C. If the difference between ABC and CBA is 297 and the difference between ABC and BAC is 450, find all such possible three digit numbers ABC and find their sum.

Solution Given that

$$(100A+10B+C)-(100C+10B+A)=99(A-C)=297$$

Hence $A-C=3$. Similarly, from

$$(100A+10B+C)-(100B+10A+C)=90(A-B)=450$$

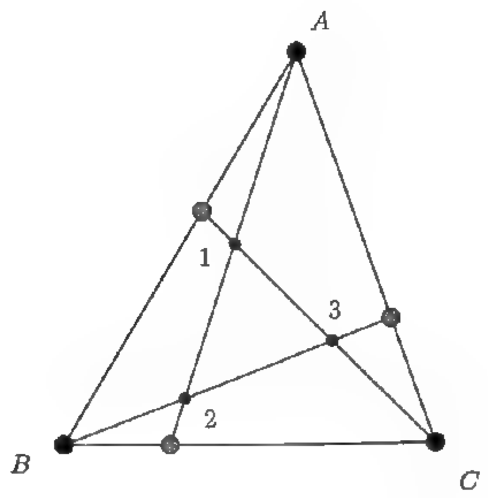
we have $A-B=5$. The possible values are given below:

A	B	C	Number ABC
5	0	2	502
6	1	3	613
7	2	4	724
8	3	5	835
9	4	6	946

Thus there are five numbers and their sum is 3620.

5. Take a triangle. Three straight lines are drawn through its vertices as shown. The maximum number of points of intersection of these three lines is 3.
If we draw two lines through each vertex of the triangle, what is the maximum number of points of intersection of these 6 lines? What if we draw three lines through each vertex? Can you guess what will be the maximum number of intersections if we draw 4 lines through each vertex?

Solution Consider the line through A. This is cut in



two points by the lines from B and C. Thus on this line there are two points of intersection. Similarly line from B has two points of intersection by the lines from A and C. Again line from C has two points of intersection cut by the lines from A and B. Thus we have six points of intersection. But, each point of intersection is counted twice – for example, the point of intersection 1 is counted once as lying on the line from A and again as lying on the line from C. Thus there are only three points of intersection. This simple *counting technique* helps to count the points of intersection in the general case as well.

Let us take the case of two lines through each vertex. Consider a line from A. This will have two pints of intersection on it induced by the lines from B and again two points from the lines from C. Thus it has 4 points of intersection. Each line thus has 4 points on it. Since we have 6 lines totally, we have $6 \times 4 = 24$ points. But as observed above, we have counted each intersection twice. Thus there are 12 points of intersection in this case.

If we draw three lines, then we have a total of 9 lines and each line has 6 intersections. Thus the total number

of intersections is $(9 \times 6)/2 = 27$. In the case of four lines through each vertex, we have $(12 \times 8)/2 = 48$ points of intersection. More generally, is there are n lines through each vertex, then the maximum number of points of intersection is $(3n \times 2n)/2 = 3n^2$.

FINAL – KAPREKAR CONTEST
NMTC at JUNIOR LEVEL
VII & VIII Standards

1. (a) The diagram below contains 13 boxes. The numbers in the second and twelfth boxes are respectively 175 and 70. Fill up the boxes with natural numbers such that

   i. sum of all numbers in all the 13 boxes is 2015

   ii. sum of the numbers in any three consecutive boxes is always the same

	175										70	

   (b) If x, y, z are real and unequal numbers, prove that

$$2015x^2 + 2015y^2 + 6z^2 > 2(2012xy + 3yz + 3zx)$$

Solution

(a) If the number in the box i is x_i, then we have

$$x_i + x_{i+1} + x_{i+2} = x_{i+1} + x_{i+2} + x_{i+3}$$

and hence $x_i = x_{i+3}$ for all i. Thus the entries in the boxes are $x_1, x_2, x_3, x_1, x_2, x_3, \ldots, x_3, x_1$. Since $x_3 = x_{12} = 70$ and $4(x_1 + x_2 + x_3) + x_1 = 2015$, it follows that $x_1 = 207$. Hence the numbers in the boxes are
$207, 175, 70, 207, 175, 70, 207, 175, 70, 207, 175, 70, 207$.

(b) Since $(x - y)^2 > 0$ when $x \neq y$, and similar inequalities for y, z and z, x, we have

$$x^2 + y^2 > 2xy \tag{1}$$
$$y^2 + z^2 > 2yz \tag{2}$$
$$z^2 + x^2 > 2zx \tag{3}$$

Multiply (1) by 2012, (2) and (3) by 3 and add to get

$$2015x^2 + 2015y^2 + 6z^2 > 2(2012xy + 3yz + 3zx)$$

the desired inequality.

2. Find a, b, c if they are real numbers, $a + b = 4$ and $2c^2 - ab = 4\sqrt{3}c - 10$.
Solution

$$\begin{aligned} 2c^2 - 4\sqrt{3}c - 10 = ab \\ &= \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 \\ &- 4 - \left(\frac{a-b}{2}\right)^2 \end{aligned}$$

Hence

$$\begin{aligned} 2(c^2 - 2\sqrt{3}c - 3) + \left(\frac{a-b}{2}\right)^2 &= 0 \\ 2(c - \sqrt{3})^2 + \left(\frac{a-b}{2}\right)^2 &= 0 \end{aligned}$$

Since a, b, c are real, $c = \sqrt{3}$ and $a = b = 2$.

3. When $a = 2^{2014}$ and $b = 2^{2015}$, prove that

$$\left\{\frac{\frac{(a+b)^2+(a-b)^2}{b-a} - (a+b)}{\frac{1}{b-a} - \frac{1}{a+b}}\right\} \div \left\{\frac{(a+b)^3 + (b-a)^3}{(a+b)^2 - (a-b)^2}\right\}$$

is divisible by 3.
Solution Put $a + b = x$ and $b - a = y$. The given

expression is

$$\left\{\frac{\frac{x^2+y^2}{y}-x}{\frac{1}{y}-\frac{1}{x}}\right\}\times\left\{\frac{x^2-y^2}{x^3+y^3}\right\}$$

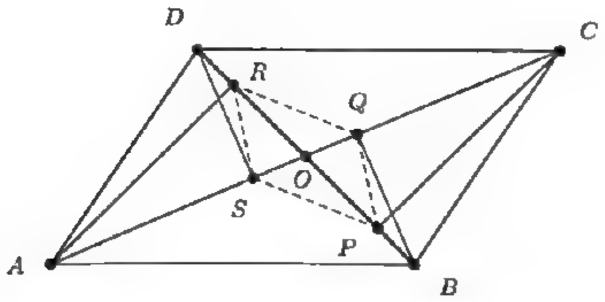
$$=\frac{x(x^2+y^2-xy)}{x-y}\times\frac{(x-y)(x+y)}{(x+y)(x^2+y^2-xy)}$$

$$=x=a+b=2^{2014}+2^{2015}$$

$$=2^{2014}(1+2)=3\times 2^{2104}$$

4. Prove that the feet of the perpendiculars drawn from the vertices of a parallelogram onto its diagonals are the vertices of another parallelogram.

Solution Let the diagonals of the given parallelogram



$ABCD$ intersect at O and P, Q, R, S the feet of the perpendiculars from the vertices on the diagonals. In triangles OSD and OQB, we have

$\angle OSD = \angle OQB = 90^\circ$

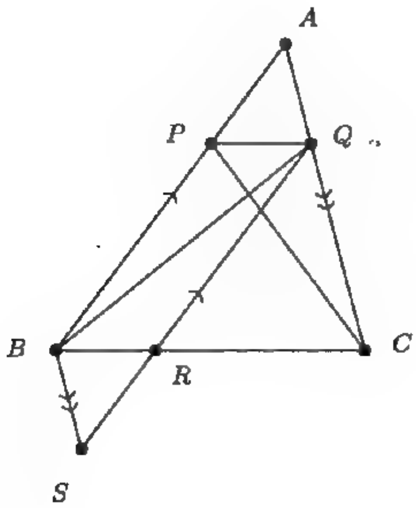
$\angle SOD = \angle BOQ$ (vertically opposite angles)

$OD = OB$ (in $ABCD$ diagonals bisect each other)

Thus triangles OSD and OQB are congruent and hence $OS = OQ$. Similarly, triangles OAR and OCP are congruent and $OR = OP$. Thus in $PQRS$ diagonals bisect each other and consequently, $PQRS$ is a parallelogram.

5. ABC is an acute angled triangle. P, Q are points on AB and AC respectively such that the area of $\triangle APC =$ area of $\triangle AQB$. A line is drawn through B parallel to AC and meets the line through Q parallel to AB at S. QS cuts BC at R. Prove that $RS = AP$.

Solution Given Area $\triangle APC =$ Area $\triangle AQB$. Since



$$\text{Area } \triangle APC = \text{Area } \triangle APQ + \text{Area } \triangle PCQ$$

and

$$\text{Area } \triangle AQB = \text{Area } \triangle APQ + \text{Area } \triangle PQB$$

it follows that

$$\text{Area } \triangle PCQ = \text{Area } \triangle PQB$$

Since these have the same base PQ, it follows that PQ is parallel to BC. Thus $PQRB$ is a parallelogram and $QR = PB$. Also $ABSQ$ is a parallelogram and $AB = QS$. Hence we have

$$AP = AB - PB = QS - PB = QS - QR = RS$$

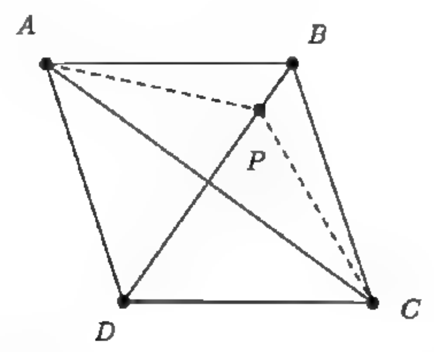
6. (a) A man is walking from a town A to another town B at a speed of 4 kms per hour. A bus started from town A one hour later and is travelling at a speed of 12 kms per hour. The man on the way got into the bus when it reached him and travelled further two hours to reach the town B. What is the distance between the towns A and B?

   (b) A point P is taken within a rhombus $ABCD$ such that $PA = PC$. Show that B, P, D are collinear.

Solution

(a) Suppose that the man has walked x kms before the bus reached him. Then we have $\frac{x}{4} - 1 = \frac{x}{12}$ and hence $x = 6$. After boarding the bus, he travels for 2 hours at the speed of 12 kms per hour and hence travels 24 kms. Thus the distance between A and B is $6 + 24 = 30$ kms.

(b) In a rhombus diagonals bisect each other at right angles. Hence BD is the perpendicular bisector of AC. Since $PA = PC$, it follows that P lies on the



perpendicular bisector of AC and hence on BD. Thus B, P, D are collinear.

7. If

$$(x+y+z)^3 = (y+z-x)^3+(z+x-y)^3+(x+y-z)^3+kxyz$$

find the numerical value of k.
If $a = 2015, b = 2014, c = \frac{1}{2014}$, prove that

$$(a+b+c)^3 - (a+b-c)^3 \\ - (b+c-a)^3 - (c+a-b)^3 - 23abc = 2015$$

Solution Putting $x = y = z = 1$ we get $27 = 1+1+1+k$ and hence $k = 24$. Also

$$(a+b+c)^3 - (a+b-c)^3 \\ - (b+c-a)^3 - (c+a-b)^3 - 23abc = abc = 2015$$

FINAL – BHASKARA CONTEST
NMTC at JUNIOR LEVEL
IX & X Standards

1. (a) 28 integers are chosen from the interval $[104, 208]$. Show that there exist two of them with a common prime divisor.

   (b) C is a point on the line segment AB. $ACPQ$ and $CBRS$ are squares drawn on the same side of AB. Prove that S is the orthocenter of the triangle APB.

Solution

(a) The given interval has 105 integers. Let A be the subset of integers in the interval $[104, 208]$ that are divisible by at least one of the primes $2, 3, 5, 7$ and B, the complement of A. Let us find how many elements are in A.
For a positive integer d, let N_d denote the set of integers in the given interval that are divisible by d and n_d the number of elements in N_d. Clearly $A = N_2 \cup N_3 \cup N_5 \cup N_7$. Note that $n_d = \left[\frac{208}{d}\right] - \left[\frac{103}{d}\right]$ where for any real number x, $[x]$ denotes the integer part of x. We have

$$n_2 = \left[\frac{208}{2}\right] - \left[\frac{103}{2}\right] = 104 - 51 = 53$$
$$n_3 = \left[\frac{208}{3}\right] - \left[\frac{103}{3}\right] = 69 - 34 = 35$$
$$n_5 = \left[\frac{208}{5}\right] - \left[\frac{103}{5}\right] = 41 - 20 = 21$$
$$n_7 = \left[\frac{208}{7}\right] - \left[\frac{103}{7}\right] = 29 - 14 = 15$$

Similarly, we find

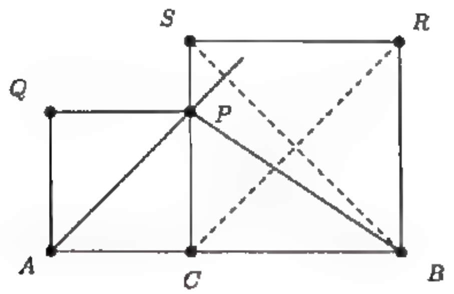
$$n_6 = 17, \quad n_{10} = 10, \quad n_{14} = 7, \quad n_{15} = 7, \quad n_{21} = 5,$$
$$n_{35} = 3, \quad n_{105} = 1, \quad n_{70} = 1, \quad n_{42} = 2, \quad n_{30} = 3,$$

and $n_{210} = 0$. By inclusion exclusion formula, the number of elements in the set A is given by

$$\begin{aligned} |A| &= n_2 + n_3 + n_5 + n_7 \\ &\quad - (n_6 + n_{10} + n_{14} + n_{15} + n_{21} + n_{35}) \\ &\quad + (n_{105} + n_{70} + n_{42} + n_{30}) \\ &\quad - n_{210} \\ &= 82 \end{aligned}$$

Thus B contains $105 - 82 = 23$ elements. When we select 28 integers, at least 5 integers should be from the set A. Each of these 5 is divisible by at least one of the four primes. Hence by pigeon hole principle, at least two must have a common prime divisor among $2, 3, 5, 7$.

(b) The diagonals AP and CR of the squares $ACPQ$ and $CBRS$ are parallel. Also, in the square $CBRS$, the diagonals CR and BS are perpendicular. Hence BS is the altitude of the



triangle APB through the vertex B. Also SC being perpendicular to AB, it is the altitude of

the triangle APB through the vertex P. These two altitudes intersect at S and hence S is the orthocenter of the triangle APB.

2. (a) a, b, c are distinct real numbers such that

$$a^3 = 3(b^2 + c^2) - 25$$
$$b^3 = 3(c^2 + a^2) - 25$$
$$c^3 = 3(a^2 + b^2) - 25$$

Find abc.

(b) Let

$$a = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{2015^2}$$

Find the largest integer $\leq a$.

Solution

(a) Let $x^3 - px^2 + qx - r = 0$ be the equation whose roots are a, b, c. Then

$$p = a + b + c, \qquad q = ab + bc + ca, \qquad r = abc$$

We have

$$\begin{aligned} a^3 &= 3(a^2 + b^2 + c^2) - 3a^2 - 25 \\ &= 3(p^2 - 2q) - 3a^2 - 25 \end{aligned}$$

Similarly,

$$b^3 = 3(p^2 - 2q) - 3b^2 - 25$$
$$c^3 = 3(p^2 - 2q) - 3c^2 - 25$$

From the above, it follows that the roots of the equation

$$x^3 + 3x^2 - (3(p^2 - 2q) - 25) = 0$$

are also a, b, c.
Comparing this with $x^3 - px^2 + qx - r = 0$, we have

$$p = -3, \qquad q = 0, \qquad r = 3(p^2 - 2q) - 25$$

Thus $abc = r = 3((-3)^2) - 2 \times 0) - 25 = 2$.

(b)

$$\begin{aligned} a &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{2015^2} \\ &< 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{2014 \cdot 2015} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \cdots + \left(\frac{1}{2014} - \frac{1}{2015}\right) \\ &= 1 + 1 - \frac{1}{2015} = 1\frac{2014}{2015} \end{aligned}$$

Since $a \geq 1$, the largest integer $\leq a$ is 1.

3. The arithmetic mean of a number of pairwise distinct primes is 27. Determine the biggest prime among them.
Solution Let the primes be $p_1 < p_2 < \cdots < p_n$. We have

$$\frac{p_1 + p_2 + \cdots + p_n}{n} = 27 \Rightarrow p_1 + p_2 + \cdots + p_n = 27n$$

The primes less than 27 are $2, 3, 5, 7, 11, 13, 17, 19, 23$.

$$\begin{aligned} p_n &= 27n - (p_1 + p_2 + \cdots + p_{n-1}) \\ &= 27 + (27 - p_1) + (27 - p_2) + \cdots + (27 - p_{n-1}) \\ &\leq 27 + (27 - 2) + (27 - 3) + \cdots + (27 - 23)) \\ &= 145 \end{aligned}$$

Since the largest prime less than 145 is 139, we have $p_n = 139$. Also when the arithmetic mean of $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31$ and 139 is 27. Thus the largest prime is 139.

4. Sixty five bugs are placed at different squares of a 9×9 board. A bug in each square moves to a horizontal or vertical adjacent square. No bug makes two horizontal or two vertical moves in succession. Show that after some moves, there will be at least two bugs in the same square.
Solution Let the square in the i-th row and j-th column be denoted by (i, j). We have $1 \leq i, j \leq 9$.
Let us define type of a square as follows:

   (a) Type A: square (i, j) for which i, j are both odd

   (b) Type B: square (i, j) for which i, j are both even

   (c) Type C: the rest of the squares

   There are 25 squares of Type A, 16 squares of Type B and 40 squares of Type C. A bug in Type A square will move to a Type B square after two moves and similarly a bug in Type B square will move to a Type A square after two moves. Since there are only 16 squares of Type B, if no two bugs land in the same square at any point of time, then there can be only 32 bugs in the A and B squares. Again, since a bug in Type C square moves to either a Type A square or a Type B square after one move, it follows that there can be no more than 32 bugs in Type C squares. Thus if no two bugs land in the same square, there can be no more than 64 bugs. But since there are 65 bugs, it follows that two bugs must land in the same square after some moves.

5. $f(x)$ is a fifth degree polynomial. Given that $f(x)+1$ is divisible by $(x-1)^3$ and $f(x)-1$ is divisible by $(x+1)^3$, find $f(x)$.
Solution Since $f(x)+1$ is divisible by $(x-1)^3$, it follows that $f(-x)+1$ is divisible by $(-x-1)^3$ and hence by $(x+1)^3$. Thus $(x+\text{-}1)^3$ divides both $f(x)-1$ and $f(-x)+1$ and hence divides their sum $f(x)+f(-x)$.

Similarly, $(x-1)^3$ also divides $f(x)+f(-x)$. But since $f(x)$ is a polynomial of degree 5, $f(x)+f(-x) \equiv 0$. Thus $f(x)$ contains only odd powers of x. Since $f(x)+1$ is divisible by $(x-1)^3$,

$$f(x)+1=(x-1)^3(Ax^2+Bx-1)$$

Equating the even degree terms to zero, we get

$$B-3A=0, \quad -A+3B+3=0$$

Solving for A, B, we get $A=-\frac{3}{8}$, $B=-\frac{9}{8}$. Thus

$$\begin{aligned} f(x) &= (x^3-3x^2+3x-1)\left(-\frac{3}{8}x^2-\frac{9}{8}x-1\right)-1 \\ &= -\frac{3}{8}x^5+\frac{5}{4}x^3-\frac{15}{8}x \end{aligned}$$

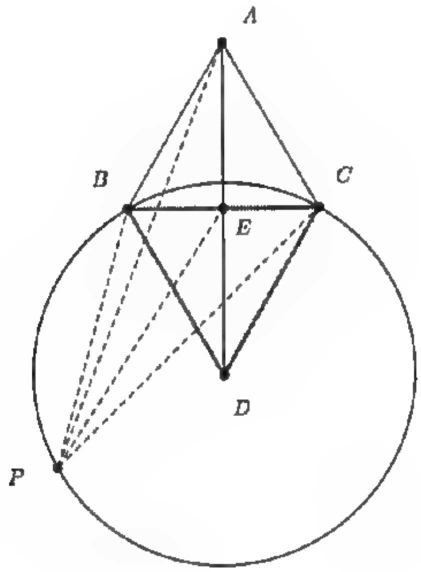
6. ABC and DBC are two equilateral triangles on the same base BC. A point P is taken on the circle with center D and radius BD. Show that PA, PB, PC are the sides of a right angled triangle.

Solution 1 Clearly $ABDC$ is a rhombus. Let E be the mid point of BC. The diagonals AD and BC bisect each other at right angles at E. In triangle APD, by Apollonius theorem, we have

$$\begin{aligned} PA^2+PD^2 &= 2PE^2+2DE^2 \\ &= 2PE^2+2(BD^2-BE^2) \end{aligned}$$

Thus

$$\begin{aligned} PA^2 &= 2PE^2+2(BD^2-BE^2)-PD^2 \\ &= 2PE^2+BD^2-2BE^2 \qquad \text{since } PD=BD \\ &= 2PE^2+4BE^2-2BE^2 \\ &= 2PE^2+2BE^2 \\ &= PB^2+PC^2 \qquad \text{from triangle } PBC \end{aligned}$$



Thus PA, PB, PC are the sides of a right angled triangle.

Solution 2 There is no loss of generality in assuming that D is the origin, $BD = 1$ and DA is the $Y-$axis. The point B has coordinates $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Similarly, C is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $A(0, \sqrt{3})$. Let $P(\cos\theta, \sin\theta)$ be any point on the circle. We have

$$PA^2 = \cos^2\theta + (\sin\theta - \sqrt{3})^2 + \sin^2\theta = 4 - 2\sqrt{3}\sin\theta$$

$$PB^2 = \left(\cos\theta + \frac{1}{2}\right)^2 + \left(\sin\theta - \frac{\sqrt{3}}{2}\right)^2$$

$$PC^2 = \left(\cos\theta - \frac{1}{2}\right)^2 + \left(\sin\theta - \frac{\sqrt{3}}{2}\right)^2$$

Now,

$$\begin{aligned} PB^2 + PC^2 &= 2\left(\cos^2\theta + \frac{1}{4}\right) + 2\left(\sin^2\theta - \sqrt{3}\sin\theta + \frac{3}{4}\right) \\ &= 4 - 2\sqrt{3}\sin\theta = PA^2 \end{aligned}$$

7. a, b, c are real numbers such that $a+b+c=0$ and $a^2+b^2+c^2=1$. Prove that $a^2b^2c^2 \leq \dfrac{1}{54}$. When does the equality hold?

Solution If one of a, b or c is zero, the result is trivial. Assume that $abc \neq 0$. Let us first consider the case $a, b > 0$. Here, $c = -(a+b)$. Also,

$$a^2+b^2+c^2 = a^2+b^2+(a+b)^2 = 2(a^2+b^2+ab) = 1$$

Thus $a^2+b^2+ab = \frac{1}{2}$. Also since $a, b > 0$, by AM - GM inequality, $2ab \leq a^2+b^2$. Thus

$$3ab = ab + 2ab \leq ab + a^2 + b^2 = \frac{1}{2}$$

Hence $ab \leq \dfrac{1}{6}$. Now,

$$\begin{aligned} c^2 = a^2+b^2+2ab &= (a^2+b^2+ab)+ab \\ &\leq \frac{1}{2} + + \frac{1}{6} = \frac{2}{3} \end{aligned} \tag{1}$$

Thus

$$a^2b^2c^2 \leq \frac{1}{36} \times \frac{2}{3} = \frac{1}{54}$$

Equality holds when $a = b = \dfrac{1}{\sqrt{6}}$ and $c = -\dfrac{2}{\sqrt{6}}$.

If two of the numbers are negative and the other positive, we can apply the above argument to $-a, -b, -c$ to obtain the same inequality.

FINAL – RAMANUJAN CONTEST

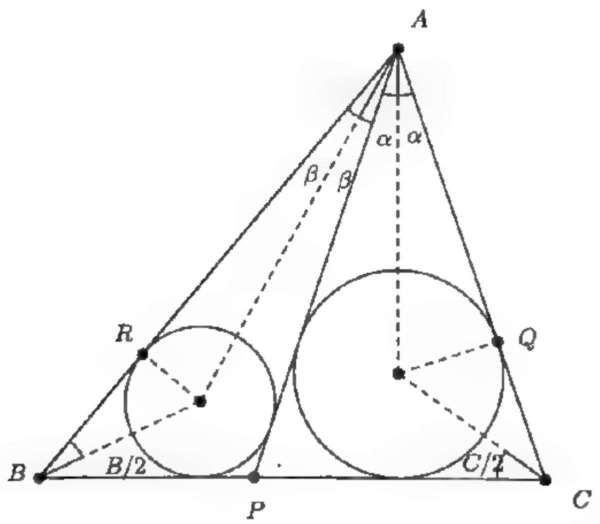
NMTC at INTER LEVEL

XI & XII Standards

1. P is a point on the base BC of a triangle ABC. r, r_2, r_3 are the inradii of the triangles ABC, ACP, ABP respectively. h_a is the altitude of the triangle ABC from A. Prove that

$$\frac{1}{r_2}+\frac{1}{r_3}-\frac{r}{r_2r_3}=\frac{2}{h_a}$$

Solution Let $BC = a, CA = b, AB = c$. Let α, β be



the angles indicated in the Figure. We have

$$\cot\alpha = \frac{AQ}{r_2} = \frac{AC-CQ}{r_2} = \frac{b}{r_2} - \cot\frac{C}{2}$$
$$\cot\beta = \frac{AR}{r_3} = \frac{AB-BR}{r_3} = \frac{c}{r_3} - \cot\frac{B}{2}$$

Also

$$\cot\frac{A}{2} = \cot(\alpha+\beta) = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha+\cot\beta}$$

and hence

$$\cot\frac{A}{2}\left(\frac{b}{r_2}-\cot\frac{C}{2}+\frac{c}{r_3}-\cot\frac{B}{2}\right)$$
$$=\left(\frac{b}{r_2}-\cot\frac{C}{2}\right)\left(\frac{c}{r_3}-\cot\frac{B}{2}\right)-1$$

Rearranging, we get

$$\frac{b}{r_2}\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)+\frac{c}{r_3}\left(\cot\frac{A}{2}+\cot\frac{C}{2}\right)-\frac{bc}{r_2r_3}$$
$$=\cot\frac{A}{2}\cot\frac{B}{2}+\cot\frac{B}{2}\cot\frac{C}{2}+\cot\frac{C}{2}\cot\frac{A}{2}-1 \quad (1)$$

Since

$$\cot\frac{A}{2}=\frac{s-a}{r},\quad \cot\frac{B}{2}=\frac{s-b}{r},\quad \cot\frac{C}{2}=\frac{s-c}{r}$$

we have

$$\left(\sum\cot\frac{A}{2}\cot\frac{B}{2}\right)-1=\sum\frac{(s-a)(s-b)}{r^2}-1$$
$$=\frac{1}{r^2}(3s^2-2s(a+b+c)$$
$$+ab+bc+ca)-1$$
$$=\frac{1}{r^2}(ab+bc+ca-s^2)-1$$

Since area Δ of the triangle is given by

$$\Delta=rs=\frac{abc}{4R}=\sqrt{s(s-a)(s-b)(s-c)}$$

we have

$$r^2s^2=s(s-a)(s-b)(s-c)$$
$$\Rightarrow r^2s=s^3-s^2(a+b+c)+s(ab+bc+ca)-abc$$
$$=s(ab+bc+ca-s^2)-abc$$
$$\Rightarrow \frac{1}{r^2}(ab+bc+ca-s^2)=\frac{4R\Delta}{r\Delta}+1$$

Hence

$$\left(\sum \cot\frac{A}{2}\cot\frac{B}{2}\right) - 1 = \frac{4R}{r}$$

Again

$$b = r\left(\cot\frac{A}{2} + \cot\frac{C}{2}\right), \quad c = r\left(\cot\frac{A}{2} + \cot\frac{B}{2}\right)$$

Thus from (1), we get

$$\frac{bc}{rr_2} + \frac{bc}{rr_3} - \frac{bc}{r_2r_3} = \frac{4R}{r}$$

This gives

$$\frac{1}{r_2} + \frac{1}{r_3} - \frac{r}{r_2r_3} = \frac{4R}{bc} = \frac{4Ra}{abc} = \frac{a}{\Delta} = \frac{a}{\frac{1}{2}a\cdot h_a} = \frac{2}{h_a}$$

2. (a) If

$$x = \left(b^{\frac{2015}{2016}} - a^{\frac{2015}{2016}}\right)^{\frac{2016}{2015}}$$

find the value of

$$\sqrt[2015]{x^{2015} + \sqrt[2016]{a^{2015}x^{(2015)^2}}} + \sqrt[2015]{a^{2015} + \sqrt[2016]{x^{2015}a^{(2015)^2}}} - b$$

(b) If

$$N = \sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} + \sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} + \cdots + \sqrt{1 + \frac{1}{2014^2} + \frac{1}{2015^2}}$$

find $[N]$, the integral part of N.

Solution

(a) Let $p = 2015$ and $q = 2016$. We have $q = p + 1$. We have

$$x^{p/q} = b^{p/q} - a^{p/q}$$

Now,

$$\begin{aligned}
x^p + \sqrt[q]{a^p x^{p^2}} &= x^p + a^{p/q} x^{p^2/q} \\
&= x^p + \left(b^{p/q} - x^{p/q}\right) x^{p^2/q} \\
&= x^p + b^{p/q} x^{p^2/q} - x^{p/q + p^2/q} \\
&= x^p + b^{p/q} x^{p^2} q - x^{(p/q)(1+p)} \\
&= x^p + b^{p/q} x^{p^2/q} - x^p \text{ since } p + 1 = q \\
&= b^{p/q} x^{p^2/q}
\end{aligned}$$

Similarly,

$$a^p + \sqrt[q]{x^p a^{p^2}} = b^{p/q} a^{p^2/q}$$

Hence the given expression equals

$$\begin{aligned}
&\left(b^{p/q} x^{p^2/q}\right)^{1/p} + \left(b^{p/q} a^{p^2/q}\right)^{1/p} - b \\
&= b^{1/q}\left(x^{p/q} + a^{p/q}\right) - b \\
&= b^{1/q} b^{p/q} - b = b^{(p+1)/q} - b \\
&= b - b = 0
\end{aligned}$$

(b) Let t_n denote the n-th term.

$$\begin{aligned}
t_n &= \sqrt{1+\frac{1}{n^2}+\frac{1}{(n+1)^2}} \\
&= \sqrt{\left(1+\frac{1}{n}\right)^2-\frac{2}{n}+\frac{1}{(n+1)^2}} \\
&= \sqrt{\left(\frac{n+1}{n}\right)^2-\frac{2}{n}+\frac{1}{(n+1)^2}} \\
&= \sqrt{\left(\frac{n+1}{n}-\frac{1}{n+1}\right)^2} \\
&= \sqrt{\left(1+\frac{1}{n}-\frac{1}{n+1}\right)^2} \\
&= 1+\frac{1}{n}-\frac{1}{n+1}
\end{aligned}$$

Hence

$$\begin{aligned}
N &= \left(1+\frac{1}{1}-\frac{1}{2}\right)+\left(1+\frac{1}{2}-\frac{1}{3}\right) \\
&\quad +\cdots+\left(1+\frac{1}{2014}-\frac{1}{2015}\right) \\
&= 2015-\frac{1}{2015}
\end{aligned}$$

Hence $[N]=2014$.

3. (a) a,b,c are nonzero real numbers such that $a+b+c=abc$ and $a^2=bc$. Prove that $a^2 \geq 3$.

(b) Find all prime numbers x,y,z such that

$$x(x+y)=z+120$$

Solution

(a) We have $b+c=abc-a=a^3-a$ and $bc=a^2$. Thus b,c are the roots of the equation

$$t^2-(a^3-a)t+a^2=0$$

Since b, c are real, the discriminant of this quadratic ≥ 0. Thus

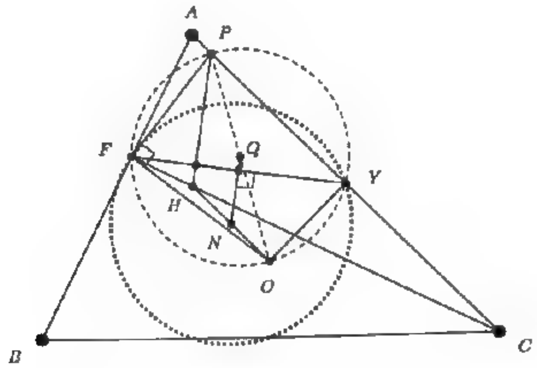
$$\begin{aligned}&(a^3-a)^2-4a^2\geq 0\\ \Rightarrow\ &a^2(a^2+1)(a^2-3)\geq 0\\ \Rightarrow\ &a^2\geq 3\end{aligned}$$

(b) If $z=2$, then we have $x(x+y)=122$. Since x is a prime and $2, 61$ are the only factors of 122, it follows that $x=2$ or $x=61$. But if $x=61$ then $x+y=2$, and y becomes negative. Thus $x=2, y=59, z=2$ is one solution.
If z is odd, $x, x+y$ are both odd and hence y is even. Thus $y=2$. In this case, we have

$$x(x+2)=z+122\Rightarrow(x-10)(x+12)=z$$

Since z is a prime, it follows that $x-10=1$ and $z=23$. Thus $x=11, y=2, z=13$ is the other solution.

4. Let ABC be an acute angled triangle with $BC>AC$. Let O be the circumcenter and H, the orthocenter of the triangle ABC. F is the foot of the perpendicular from C on AB and the perpendicular to OF at F meets the side CA at P. Show that $\angle FHP=\angle A$.
Solution Let Y be the midpoint of AC. Clearly, $OY\perp AC$. Since $\angle OFP=90°$, it follows that the quadrilateral $OFPY$ is cyclic. The center of the circumcircle of $OFPY$ is the midpoint Q of OP. Let N be the nine point center of the triangle ABC. Then N is the midpoint of OH and the nine point circle passes through F and Y. Thus the line NQ is the line joining the centers of these two circles and FY is their common chord. Hence $NQ\perp FY$. Also from the triangle OPH, since $OQ=QP, ON=NH$, it follows that $NQ\parallel HP$



and thus $HP \perp FY$. Since $CF \perp AB$, it follows that the angle between HP and CF is equal to the angle between their perpendiculars FY and AB. Thus $\angle FHP = \angle YFA$. But since Y is the midpoint of CA and the perpendicular from Y on AB is parallel to CF, it also bisects AF. Thus $\angle YFA = \angle YAF = \angle A$.

5. Is it possible to write the numbers $1, 2, 3, \ldots, 121$ in a 11×11 table so that any two consecutive numbers be written in cells with a common side and all perfect squares lie in a single column?

Solution Let us assume that such a table exists. The table would be divided into two parts by the single column containing the perfect squares with one side containing $11n\,(0 \le n \le 5)$ numbers and the rest $110 - 11n$ numbers on the other side. Note that the numbers between any two perfect squares a^2 and $(a+1)^2$ must lie on one side since they can not cross over the perfect square column and those between $(a+1)^2$ and $(a+2)^2$ lie on the other side. Now the number of integers that lie strictly between $1^2, 2^2, 3^2, \ldots, 11^2$ are $2, 4, 6, 8, 10, 12, 14, 16, 18, 20$ respectively. So one side must have $2+6+10+14+18 = 50$ numbers and the other side must have $110 - 50 = 60$ numbers. Since neither of these is a multiple of 11, it follows that such a table does

not exist.

6. (a) The positive integers are separated into two subsets with no common elements. Show that one of these two subsets must contain a three term arithmetic progression.

   (b) If two rays through the vertex of an angle make equal angles with the sides, they are said to be isogonal. Clearly, an angle bisector is isogonal to itself. If AM, BN, CP through the vertices A, B, C of a triangle are concurrent, show that the isogonals AM', BN', CP' are also concurrent.

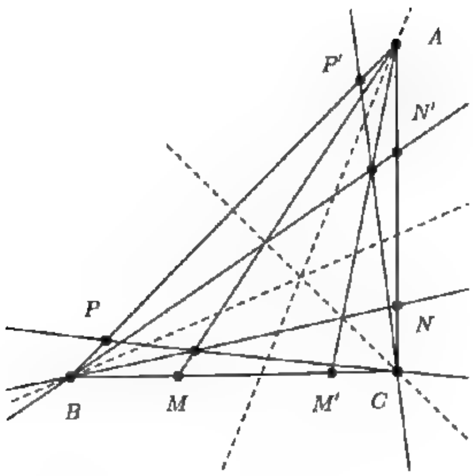
Solution

(a) Let x be an integer greater than 6.
If $x+2, x+4, x+6$ are in the same subset, the problem is solved. Otherwise, x and at least one of $x+2, x+4, x+6$, say $x+2y$ are in the same subset. If this set also contains one of $x-2y, x+y, x+4y$, then again the problem is solved. If not, $x-2y, x+y, x+4y$ are in the other set and this is a three term arithmetic progression.

(b) A line and its isogonal make equal angles with the bisector of the angle. We recall the trigonometric form of Ceva's theorem:
Three cevians AM, BN, CP are concurrent if and only if

$$\frac{\sin\angle BAM}{\sin\angle MAC}\cdot\frac{\sin\angle CBN}{\sin\angle NBA}\cdot\frac{\sin\angle ACP}{\sin\angle PCB}=1 \qquad (1)$$

If AM, AM' are isogonals, $\angle BAM = \angle CAM'$ and $\angle M'AB = \angle MAC$ with similar equalities for the isogonals BN, BN' and CP, CP'. Thus from (1),



we get

$$\frac{\sin\angle M'AC}{\sin\angle M'AB}\cdot\frac{\sin\angle N'BA}{\sin\angle N'BC}\cdot\frac{\sin\angle P'CB}{\sin\angle P'CA}=1 \qquad (2)$$

and by Ceva's theorem, it follows that AM', BN', CP' are concurrent.

7. The distance between the circumcenter and the incenter of a triangle is d. R and r are respectively the circumradius and inradius of the triangle. Prove that $\dfrac{1}{R-d}+\dfrac{1}{R+d}=\dfrac{1}{r}$.

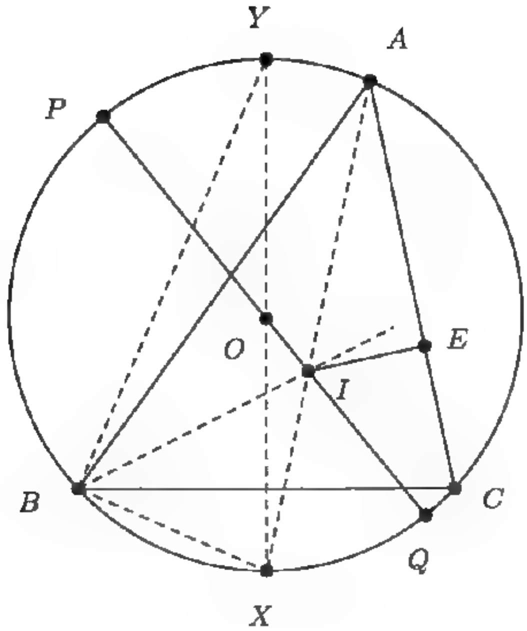
Solution

$$\frac{1}{R-d}+\frac{1}{R+d}=\frac{1}{r}\iff R^2-2Rr=d^2$$

We prove $R^2-2Rr=d^2$.

Let O be the circumcenter and I, the incenter of the triangle ABC. Let AI meet the circumcircle at X. Let XOY be the diameter of the circumcircle through X. Let OI intersect the circumcirle at P, Q.

In the right angled triangles AIE and XYB, we have $\angle IAE=\dfrac{A}{2}$, $\angle XYB=\angle XAB=\dfrac{A}{2}$. Thus the triangles are similar and $\dfrac{AI}{XY}=\dfrac{IE}{BX}$ and hence $AI\times BX=2Rr$.



Again,

$$\angle BIX = \angle IAB + \angle IBA = \frac{A}{2} + \frac{B}{2}$$

$$\angle XBI = \angle XBC + \angle CBI = \angle XAC + \frac{B}{2} = \frac{A}{2} + \frac{B}{2}$$

Thus $BX = XI$.

In the chords AIX and POQ, we have

$$PI \times IQ = AI \times IX$$

$$\Rightarrow (R + OI) \times (R - OI) = AI \times BX = 2Rr$$

Thus $OI^2 = R^2 - 2Rr$.

ARYABHATTA CONTEST
SENIOR – 2015

1. Let N be the number of positive integers that are less than or equal to 2015 and whose base 2 representation contains more 1s than 0s. Find the remainder when N is divided by 1000.

Solution For any number n such that $2^k \leq n < 2^{k+1}$, the base 2 representation contains $k+1$ digits with the leading digit 1. If

$$N = 2^k + 2^{k_1} + 2^{k_2} + \cdots + 2^{k_l}$$

where $0 \leq k_i < k$ and k_is are distinct, then N contains $l+1$ ones and $k+1-l$ zeros in its binary representation. Hence it will have more 1s than 0s if and only if $l > \left[\dfrac{k+1}{2}\right]$, where $[x]$ is the integer part of x.

N	*Count*
$1 \leq N < 2$	1
$2 \leq N < 2^2$	1
$2^2 \leq N < 2^3$	$\binom{2}{2} + \binom{2}{1} = 3$
$2^3 \leq N < 2^4$	$\binom{3}{3} + \binom{3}{2} = 4$
$2^4 \leq N < 2^5$	$\binom{4}{4} + \binom{4}{3} + \binom{4}{2} = 11$
$2^5 \leq N < 2^6$	$\binom{5}{5} + \binom{5}{4} + \binom{5}{3} = 16$
$2^6 \leq N < 2^7$	$\binom{6}{6} + \cdots + \binom{6}{3} = 42$
$2^7 \leq N < 2^8$	$\binom{7}{7} + \cdots + \binom{7}{4} = 64$
$2^8 \leq N < 2^9$	$\binom{8}{8} + \cdots + \binom{8}{4} = 163$
$2^9 \leq N < 2^{10}$	$\binom{9}{9} + \cdots + \binom{9}{5} = 256$
$2^{10} \leq N < 2^{11}$	$\binom{10}{10} + \cdots + \binom{10}{5} = 638$

Thus we have 1199 numbers between 1 and 2047 that have more 1s than 0s in their binary representation. But

we need only numbers upto 2015. Consider any number between 2016 and 2047. Since $2015 = 11111011111$, $2016 = 11111100000$ and $2047 - 2016 = 31$, all numbers between 2016 and 2047 also contain at least as many 1s as 2016 and thus contain more 1s than 0s. Thus the number of numbers between 1 and 2015 that have more 1s than 0s in their binary representation is $N = 1199 - 32 = 1167$ and $N \mod 1000 = 167$.

2. Find the average of the sums

$$|a_1 - a_2| + |a_2 \quad a_3| + \cdots + |a_{19} - a_{20}|$$

taken over all permutations $(a_1, a_2, \ldots, a_{20})$ of the integers $1, 2, \ldots, 20$.

Solution We will solve the problem for permutations of $2n$ numbers $1, 2, \ldots, 2n$.

We first note that the average of the sums is n times the average value of $|a_1 - a_2|$ since the average of $|a_{2i-1} - a_{2i}|$ is the same for all $i = 1, 2, \ldots, n$. The average value of $|a_1 - a_2|$ when $a_1 = k$ is

$$\begin{aligned} &\frac{(k-1) + (k-2)) + \cdots + 1 + 0 + 1 + \ldots + (2n-k)}{2n-1} \\ &= \frac{1}{2n-1}\left(\frac{(k-1)k}{2} + \frac{(2n-k)(2n-k+1)}{2}\right) \\ &= \frac{k^2 - (2n+1)k + n(2n+1)}{2n-1} \end{aligned}$$

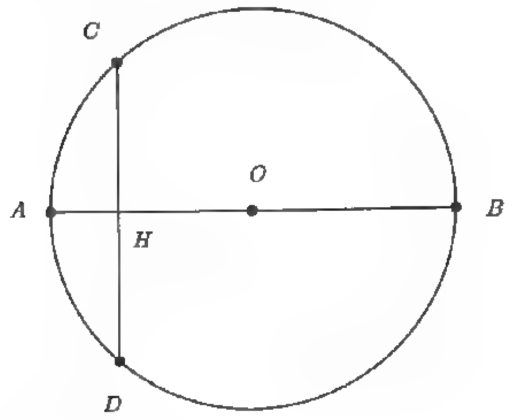
The average value of the sums is

$$n\left(\sum_{k=1}^{2n} \frac{k^2 - (2n+1)k + n(2n+1)}{2n-1}\right) = \frac{n(2n+1)}{3}$$

For $n = 10$, this average is 70.

3. The length of diameter AB of a circle is a two digit integer. Reversing the digits gives the length of the perpendicular chord CD. The distance from their intersection point H to the center O of the circle is a positive rational number. Determine the length AB.

Solution Let $AB = 10x + y$ and $CD = 10y + x$ where



x, y are positive digits.

$$OH^2 = \left(\frac{10x+y}{2}\right)^2 - \left(\frac{10y+x}{2}\right)^2$$
$$= \frac{9}{4} \times 11(x+y)(x-y)$$

Since OH is rational, $11|(x+y)(x-y)|$ must be a perfect square. Thus either $x+y$ or $x-y$ must be divisible by 11. But since x, y are digits, it follows that $x+y=11$ and $|x-y|$ must be a perfect square. Thus $x-y=\pm1,\pm9$. But clearly, $x-y=\pm9$ gives a value of 10 to x or y and hence not possible. Hence $x-y=\pm1$ and $(x,y) = (5,6)$ or $(6,5)$. But since AB is longer that CD, it follows that $x>y$ and hence $(x,y)=(6,5)$ is the only possibility and $AB = 65$.

4. Find all functions $f : [0, \infty) \to [0, \infty)$ such that

$$(f \circ f)(x) = 6x - f(x)$$

and $f(x) > 0$ for all $x > 0$.

Solution For a fixed $x \geq 0$, define the sequence $\{x_n\}_{n=0}^{\infty}$ by $x_0 = x$, $x_{n+1} = f(x_n)$, for $n \geq 1$. Then $x_n \geq 0$ for all n and

$$x_{n+2} + x_{n+1} - 6x_n = (f \circ f)(x_n) + f(x_n) - 6x_n = 0$$

Thus the sequence x_n satisfies the difference equation $x_{n+2} + x_{n+1} - 6x_n = 0$. The characteristic equation of this difference equation is $\lambda^2 + \lambda - 6 = 0$ with roots $-3, 2$. Hence the most general solution is given by

$$x_n = A2^n + B(-3)^n$$

where A, B are constants. If $B > 0$, for large odd values of n, $x_n < 0$ and if $B < 0$, for large even values of n, $x_n < 0$. Thus $B = 0$. Hence $x_n = A2^n$ and for $n = 0$, we have $x_0 = x$. Thus $A = x$. Now $f(x) = x_1 = 2x$ and this is the only function that satisfies the given conditions.

5. Let $\alpha \in (0, 2)$. Consider the sequence defined by

$$x_{n+1} = \alpha x_n + (1 - \alpha)x_{n-1}$$

for all $n \geq 1$. If $x_0 = 1, x_1 = 2$, find $\lim_{n\to\infty} x_n$ if it exists.

Solution From the given recurrence relation, we have

$$x_{n+1} - x_n = (\alpha - 1)(x_n - x_{n-1})$$

and hence

$$x_n - x_{n-1} = (\alpha - 1)^{n-1}(x_1 - x_0) = (\alpha - 1)^{n-1}$$

Now

$$\begin{aligned} x_n - x_0 &= \sum_{k=1}^{n}(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n}(\alpha-1)^{k-1} \\ &= \frac{1-(\alpha-1)^{n+1}}{1-(\alpha-1)} \end{aligned}$$

Since $\alpha \in (0,2)$, $\alpha - 1 \in (-1,1)$ and $(\alpha-1)^{n+1} \to 0$ as $n \to \infty$. Thus

$$\lim_{n\to\infty}(x_n - x_0) = \frac{1}{2-\alpha}$$

and

$$\lim_{n\to\infty} x_n = \frac{1}{2-\alpha} + x_0 = \frac{1}{2-\alpha} + 1 = \frac{3-\alpha}{2-\alpha}$$

6. Let $f : [0,1] \to \mathbb{R}$ be a continuous function with the property that $xf(y) + yf(x) \leq 1$ for all $x, y \in [0,1]$. Show that $\int_0^1 f(x)\,dx \leq \frac{\pi}{4}$.
Find a function that satisfies the given condition and for which equality holds for the above integral.
Solution Put $x = \sin t$. We have $dx = \cos t dt$ and

$$\begin{aligned} \int_0^1 f(x)dx &= \int_0^{\frac{\pi}{2}} f(\sin t)\cos t dt \\ &= \int_0^{\frac{\pi}{2}} f(\cos t)\sin t dt \\ &\qquad \text{using } \int_0^a f(u)du = \int_0^a f(a-u)du \\ &= \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} f(\sin t)\cos t dt + \int_0^{\frac{\pi}{2}} f(\cos t)\sin t dt\right) \\ &\leq \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 dt \text{ since } xf(y) + yf(x) \leq 1 \\ &= \frac{\pi}{4} \end{aligned}$$

Consider the function $f(x) = \sqrt{1-x^2}$. Since for any two positive reals a, b, we have $ab \le \frac{a^2+b^2}{2}$, we have

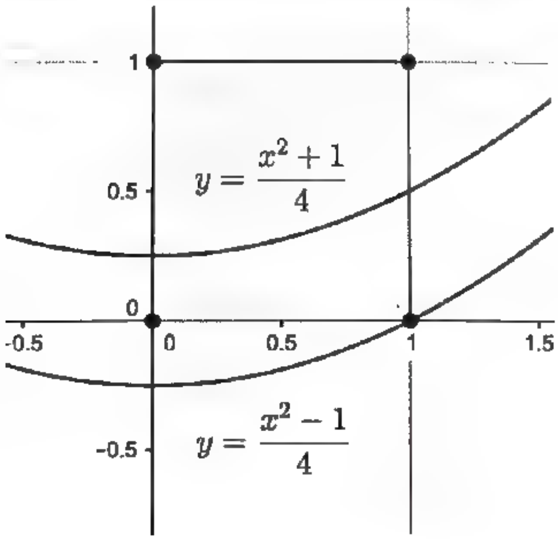
$$\begin{aligned} xf(y) + yf(x) &= x\sqrt{1-y^2} + y\sqrt{1-x^2} \\ &\le \frac{x^2+1-y^2}{2} + \frac{y^2+1-x^2}{2} \\ &= 1 \end{aligned}$$

Thus f satisfies the condition. Also,

$$\begin{aligned} \int_0^1 f(x)dx &= \int_0^1 \sqrt{1-x^2}dx \\ &= \left(\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\arcsin x\right)\Bigg|_0^1 \\ &= \frac{\pi}{4} \end{aligned}$$

7. Suppose m, n are real numbers randomly chosen in $[0, 1]$. Determine the probability that the distance between the roots of the equation $z^2 + mz + n = 0$ is not greater than 1.

Solution The roots of the equation are $\frac{-m \pm \sqrt{m^2-4n}}{2}$.



and hence the distance between them is $|m^2 - 4n|$. If

$|m^2 - 4n| \leq 1$, the point (m, n) lies in the region $-1 \leq x^2 - 4y \leq 1$. This is the region between the parabolas $4y = x^2 - 1$ and $4y = x^2 + 1$. The area of this region that lies within the sample space $[0, 1] \times [0, 1]$ is given by

$$\int_0^1 \frac{x^2 + 1}{4} dx = \frac{1}{4}\left(\frac{x^3}{3} + x\right)\Big|_0^1 = \frac{1}{3}$$

Thus the required probability is $\dfrac{1/3}{1} = \dfrac{1}{3}$.

REGIONAL MATHEMATICAL OLYMPIAD – 2015
(Tamilnadu and Pondicherry)

1. Find the maximum value of

$$\frac{(x+1/x)^6-(x^6+1/x^6)-2}{(x+1/x)^3+(x^3+1/x^3)}$$

for $x \in \mathbb{R}$ and $x > 0$.

Solution We have

$$\left[\left(x+\frac{1}{x}\right)^3+\left(x^3+\frac{1}{x^3}\right)\right]\left[\left(x+\frac{1}{x}\right)^3-\left(x^3+\frac{1}{x^3}\right)\right]$$
$$=\left(x+\frac{1}{x}\right)^6-\left(x^3+\frac{1}{x^3}\right)^2$$
$$=\left(x+\frac{1}{x}\right)^6-\left(x^6+\frac{1}{x^6}\right)-2$$

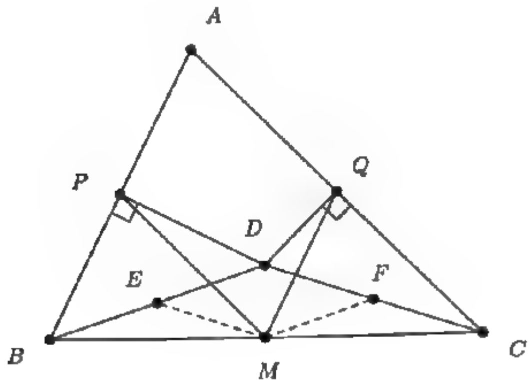
Hence the given fraction equals

$$\left(x+\frac{1}{x}\right)^3-\left(x^3+\frac{1}{x^3}\right)=3\left(x+\frac{1}{x}\right)\geq 3\times 2=6$$

and equality occurs when $x=1$.

2. P and Q are points on the sides AB and AC respectively of a triangle ABC. The perpendiculars to the sides AB and AC at P and Q respectively meet at D, an interior point of ABC. If M is the midpoint of BC, prove that $PM = QM$ if and only if $\angle BDP = \angle CDQ$.

Solution Let E and F be the midpoints of BD and CD respectively. Since $\angle BPD = \angle CQD = 90°$, it follows that $BE = ED = EP$ and $CF = FD = FQ$. Since E is the midpoint of BD and M is the midpoint



of BC, $ME \parallel CD$ and similarly $MF \parallel BD$.
Suppose that $\angle PDB = \angle CDQ = \alpha$. Consider the triangles PEM and QFM. We have $PE = ED = MF$, $QF = FD = EM$, $\angle PED = 180^\circ - 2\alpha$, $\angle QFD = 180^\circ - 2\alpha$. Since $DEMF$ is a parallelogram, $\angle DEM = \angle DFM$. Thus

$$\begin{aligned}\angle PEM &= \angle PED + \angle DEM \\ &= 180^\circ - 2\alpha + \angle DFM \\ &= \angle QFD + \angle DFM \\ &= \angle QFM\end{aligned}$$

Thus the triangles PEM and QFM are congruent and $PM = QM$.
Conversely, if $PM = QM$, then since $QF = FD = EM$ and $PE = ED = MF$, triangles PEM and QFM are congruent and $\angle PEM = \angle QFM$. Since $\angle DEM = \angle EFM$, it follows that $\angle PED = \angle QFD$. But $\angle BPD = \frac{180^\circ - \angle PED}{2}$ and $\angle QDF = \frac{180^\circ - \angle QFD}{2}$. Thus $\angle PDE = \angle QDF$.

3. Let
$$N = 2^5 + 2^{5^2} + 2^{5^3} + \cdots + 2^{5^{2015}}$$

Written in the usual decimal form, find the last two digits of N.

Solution Since $5^n - 5$ is a multiple of 5 and also since $5^n = 1 \mod 4$, it follows that $5^n - 5$ is also divisible by 4. Thus $5^n - 5$ is a multiple of 20 for all n. Also

$$\begin{aligned} 2^{20k} - 1 = 4^{10k} - 1 \\ &= (1-5)^{10k} - 1 \\ &= 1 - 10k \cdot 5 + \text{ multiples of } 25 \ - 1 \end{aligned}$$

and hence is a multiple of 25. Now for any n, we have

$$\begin{aligned} 2^{5^n} - 2^5 &= 2^5(2^{5^n - 5} - 1) \\ &= 2^5(2^{20k} - 1) \\ &= 0 \mod 100 \end{aligned}$$

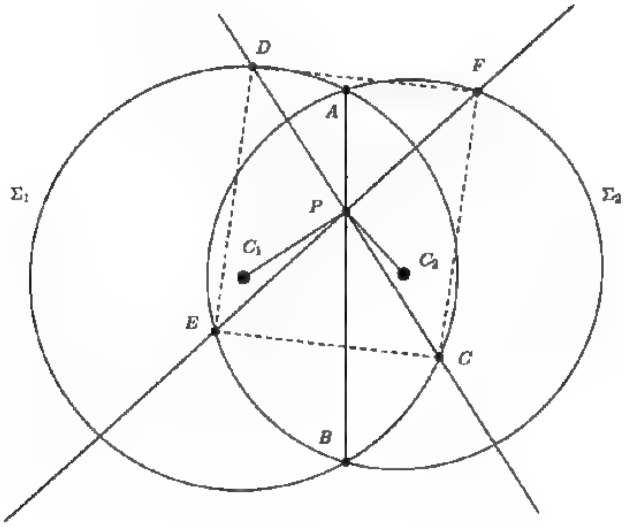
Thus $2^{5^n} = 2^5 \mod 100$ for all n. Now,

$$\begin{aligned} N &= 2^5 + 2^{5^2} + 2^{5^3} + \cdots + 2^{5^{2015}} \\ &= \underbrace{2^5 + 2^5 + \cdots + 2^5}_{\text{2015 terms}} \mod 100 \\ &= 2015 \times 32 \mod 100 \\ &= 80 \mod 100 \end{aligned}$$

Thus N ends with 80.

4. Two circles Σ_1, Σ_2 with centers C_1, C_2 intersect at A and B. Let P be any point on AB and let $AP \neq PB$. The line through P perpendicular to C_1P meets Σ_1 at C and D. The line through P perpendicular to C_2P meets Σ_2 at E and F. Show that C, D, E, F form a rectangle.

Solution Since the chord $CD \perp C_1P$, it follows that $PD = PC$. Similarly, $PE = PF$. Thus in the



quadrilateral $DECF$, diagonals bisect each other and hence it is a parallelogram.

Since AB and CD are chords of the circle Σ_1 intersecting at P, have $AP \times PB = PC \times PD$. Similarly, considering the chords AB and EF of the circle Σ_2, we have $AP \times PB = PE \times PF$. Thus $PC^2 = PE^2$ and the diagonals of the quadrilateral $DECF$ are equal and hence it is a rectangle.

5. Find all positive integers x, y such that

$$y^3 + 3y^2 + 3y = x^3 + 5x^2 - 19x + 20$$

Solution Put $u = y + 1$. We have

$$u^3 = x^3 + 5x^2 - 19x + 21$$

Thus $u^3 - x^3 = 5x^2 - 19x + 21$. The discriminant of the quadratic on the right hand side is $19^2 - 20 \times 21 < 0$ and hence it takes only positive values (since the coefficient of x^2 is positive). Thus $u^3 > x^3$ and hence $u > x$.

On the other hand, $u^3 - (x+2)^3 = -x^2 - 31x + 13$ and for $x \geq 1$, this takes only negative values. Thus $u < x + 2$.

Since u, x are integers, we have $u = x + 1$ or $y = x$. Thus

$$x^2 + 3x^2 + 3x = x^3 + 5x^2 - 19x + 20$$

and $x^2 - 11x + 10 = 0$. Thus $x = 1, 10$ and $y = x$ are the only solutions.

6. From the list of natural numbers $1, 2, 3, \ldots$, suppose we remove all multiples of 7, all multiples of 11 and all multiples of 13.

   (a) At which position in the remaining list does the number 1002 appear?

   (b) Which number occurs in position 3600?

Solution

(a) Since $7 \cdot 11 \cdot 13 = 1001$, let us first consider the list of numbers $1, 2, \ldots, 1001$ and consider how many of these remain after removing the multiples of 7, 11, 13. If N_p denotes the number of multiples of p in this list, then we have

$$N_7 = 11 \cdot 13,\ N_{11} = 7 \cdot 13,\ N_{13} = 7 \cdot 11,$$
$$N_{77} = 13,\ N_{91} = 11,\ N_{143} = 7, N_{1001} = 1$$

Thus by the principle of inclusion exclusion, the number of remaining numbers is

$$1001 - (N_7 + N_{11} + N_{13}) \\ + (N_{77} + N_{91} + N_{143}) - N_{1001} = 721$$

Hence 1002 occurs in position 722.

(b) First note that in any list of 1001 numbers

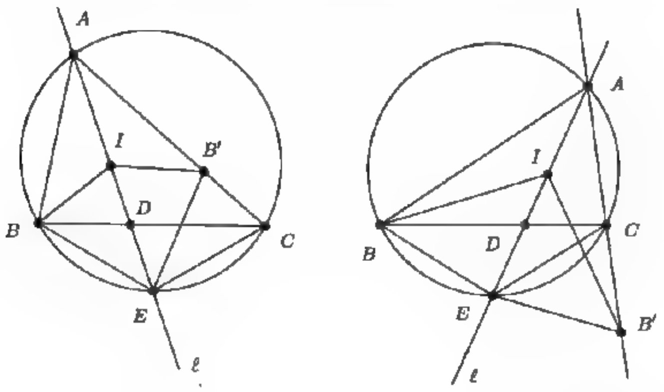
$$1001k + 1, 1001k + 2, \ldots, 1001(k + 1)$$

721 numbers will remain after the removal of multiples of 7, 11 or 13. Thus in the list of first 5×1001 numbers, $721 \times 5 = 3605$ numbers will remain. The number in position 3605 is $5 \times 1001 - 1 = 5004$. None of the numbers $5003, 5002, 5001, 5000, 4999$ is a multiple of 7, 11 or 13 and hence survive in the list. Thus the 3600 position is occupied by the number 4999.

CRMO – 2015

1. Let ABC be a triangle. Let B' denote the reflection of B in the internal bisector ℓ of $\angle A$. Show that the circumcenter of the triangle $CB'I$ lies on the line ℓ where I is the incenter of the triangle ABC.

   Solution Let ℓ meet the circumcircle of the triangle



   ABC at E. Then E is the mid point of the minor arc $\overset{\frown}{BC}$ and hence $EB = EC$. Also $\angle EBC = \angle EAC = \angle A/2$ and $\angle IBC = \angle B/2$. Hence

$$\angle BIE = \angle ABI + \angle BAI = \angle B/2 + \angle A/2$$

   Also

$$\angle IBE = \angle IBC + \angle CBE = \angle B/2 + \angle A/2$$

   Thus $\angle BIE = \angle IBE$ and hence $EB = EI$.
   Since AE is the perpendicular bisector of BB', we have $EB = EB'$. Thus we have $EB' = EC = EI$ and E is the circumcenter of the triangle $CB'I$.
   Note that the above also holds when B' lies on AC extended.

2. Let $P(x) = x^2 + ax + b$ be a quadratic polynomial where a is real and $b \neq 2$ is rational. Suppose that

$P(0)^2, P(1)^2, P(2)^2$ are integers. Prove that a and b are integers.

Solution $P(0) = b$ and since $P(0)^2$ is an integer and b is rational, it follows that b is an integer. Note that

$$P(1)^2 = (1+a+b)^2 = a^2 + 2a(1+b) + (1+b)^2 \in \mathbb{Z}$$
$$P(2)^2 = (4+2a+b)^2 = 4a^2 + 4a(4+b) + (4+b)^2 \in \mathbb{Z}$$

Eliminating a^2, we see that $4a(b-2)+4(1+b)^2-(4+b)^2 \in \mathbb{Z}$. Since $b \neq 2$, it follows that a is rational. Thus the equation

$$x^2 + 2x(1+b) + (1+b)^2 - (1+a+b)^2 = 0$$

has integer coefficients and has a rational root a. We know that if $\frac{p}{q}$ is a rational root (with p, q having no common factors other than 1) of the equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$$

where a_i are all integers, then q divides a_0 and p divides a_n. Thus if $a = \frac{p}{q}$, then q must divide the leading coefficient, 1, and hence $q = 1$. It follows that a is an integer.

3. Find all integers a, b, c such that $a^2 = bc + 4$ and $b^2 = ca + 4$.

Solution If $a = b$, we have $a^2 = ac + 4$ and hence $a(a-c) = 4$. In this case $a = \pm 1, a - c = \pm 4$, or $a = \pm 4, a - c = \pm 1$ or $a = \pm 2, a - c = \pm 2$. We obtain the solutions

$$(a,b,c) = (1,1,-3), (-1,-1,3), (4,4,3),$$
$$(-4,-4,-3), (2,2,0), (-2,-2,0)$$

If $a \neq b$, subtracting the given equations, we get $a^2 - b^2 = c(b-a)$ and hence $a+b=-c$. Substituting in the first equation, we get $a^2 = b(-a-b)+4$ or $a^2+ab+b^2=4$. The only possible solutions are

$$(a,b) = (2,0), (-2,0), (0,2), (0,-2), (2,-2), (-2,2)$$

Hence we have the solutions

$$(a,b,c) = (2,0,-2), (-2,0,2), (0,2,-2), \\ (0,-2,2), (2,-2,0), (-2,2,0)$$

4. Suppose 40 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from them so that no two of the chosen objects are adjacent or diametrically opposite?

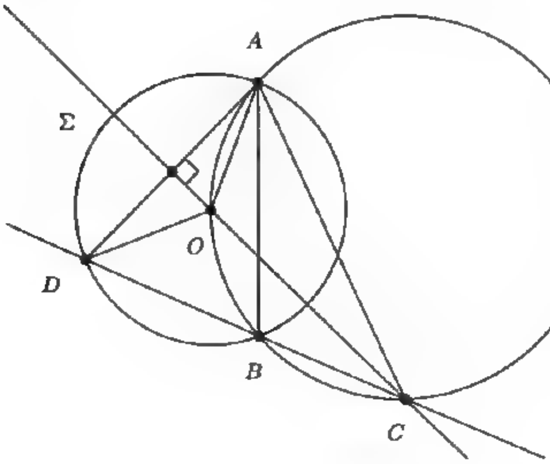
Solution Let us number the objects $1, 2, 3, \ldots, 40$. One can choose 3 out of 40 objects in $\binom{40}{3}$ ways. Among these in 40 choices all the three chosen objects are adjacent (the 40 choices are $(1,2,3), (2,3,4), \ldots, (40,1,2)$). Two are adjacent in 40×36 cases – if we choose $(i, i+1)$, we can then choose any of the 36 objects other than $i-1, i, i+1, i+2$. There are 20 diametrically opposite object pairs. When we choose one of these, we can choose any object that are neither adjacent to any of these two and hence we need to avoid 6 objects. Thus there are 20×34 choices in which two are diametrically opposite and no two are adjacent. Thus the required number of choices is

$$\binom{40}{3} - 40 - 40 \times 36 - 20 \times 34 = 7720$$

5. Two circles Γ and Σ intersect at A and B. A line through B intersects Γ and Σ again in C and D

respectively. Suppose that $CA = CD$. Show that center of Σ lies on Γ.

Solution Let the perpendicular from C to AD



intersect Γ at O. Since $CA = CD$, CO is the perpendicular bisector of AD as well as the angle bisector of the angle $\angle ACB$. Thus $OA = OD$ and $\angle OCB = \angle OCA$. Thus the chords OA, OB of Γ subtend equal angles on the circumference and hence $OA = OB$. Thus $OA = OB = OD$ and O is the circumcenter of the triangle ABD. Thus O is the center of Σ and lies on Γ.

6. How many integers m satisfy

   (a) $1 \leq m \leq 5000$

   (b) $[\sqrt{m}] = [\sqrt{m+125}]$

where for any real number x, $[x]$ denotes the largest integer not exceeding x?

Solution Let $[\sqrt{m}] = [\sqrt{m+125}] = k$. Then

$$k^2 \leq m < m + 125 < (k+1)^2$$

and hence

$$m + 125 < k^2 + 2k + 1 \leq m + 2k + 1$$

This implies $2k + 1 > 125$ and hence $k > 62$. Since $k^2 \leq m \leq 5000$, it follows that $k \leq 70$. Hence $k \in \{63, 64, \ldots, 70\}$. Observe that $63^2 = 3969$ and $64^2 = 63^2 + 127$. Hence $[63^2 + 125] = [63^2 + 1 + 125] = 63$ but $[63^2+2+125] = 64$. Thus when $k = 63$, we get two values for $m = 63^2, 63^2 + 1$. Similarly, from $65^2 = 64^2 + 129$, we get four values for $m = 64^2, 64^2 + 1, 64^2 + 2, 64^2 + 3$. Proceeding similarly, we obtain $6, 8, 10, 12, 14, 16$ values for m when $k = 65, 66, 67, 68, 69, 70$ respectively. Thus we have a total of

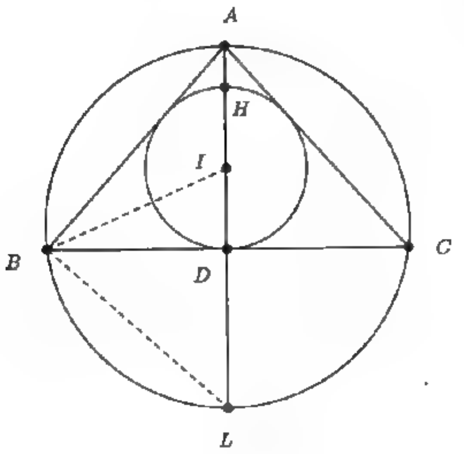
$$2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 = 72$$

values of m satisfying the given conditions.

INDIAN NATIONAL MATHEMATICAL OLYMPIAD 2016

1. Let ABC be a triangle with $AB = AC$. Suppose that the orthocenter of the triangle lies on the incircle. Find the ratio AB/BC.

Solution Let D be the midpoint of BC. Extend the



altitude AD to meet the circumcircle at L. We know that $HD = DL = 2r$, where r is the inradius. Also $\angle BIL = 90^\circ - \frac{B}{2}$ and

$$\angle IBL = \angle LBA - \angle IBA = 90^\circ - \frac{B}{2}$$

Thus $LB = LI = LD + DI = 3r$. From triangle BLD,

$$\cos \angle DLB = \sin \frac{A}{2} = \frac{LD}{LB} = \frac{2r}{3r} = \frac{2}{3}$$

Thus $\dfrac{BD}{AB} = \sin \dfrac{A}{2} = \dfrac{2}{3}$ and hence $\dfrac{BC}{AB} = \dfrac{2BD}{AB} = \dfrac{4}{3}$.

2. For positive real numbers a, b, c which of the following statements necessarily implies $a = b = c$?

(a) $a(b^3 + c^3) = b(c^3 + a^3) = c(a^3 + b^3)$

(b) $a(a^3+b^3)=b(b^3+c^3)=c(c^3+a^3)$

Solution We show that (a) need not imply $a=b=c$ whereas (b) implies $a=b=c$.
If $a(b^3+c^3)=b(c^3+a^3)$, then $c^3(a-b)=ab(a^2-b^2)$. Thus either $a=b$ or $c^3=ab(a+b)$. Similarly either $b=c$ or $a^3=bc(b+c)$.
Suppose that $a \neq b$ and $b \neq c$. Then we have

$$c^3=ab(a+b), \qquad a^3=bc(b+c)$$

Thus

$$b(a^2-c^2)+b^2(a-c)=c^3-a^3$$

If $a \neq c$, then

$$b(a+c)+b^2=-a^2-ac-c^2 \Rightarrow a^2+b^2+c^2+ab+bc+ca=0$$

a contradiction, since a,b,c are positive real numbers. Thus $a=c$ and $a^3=ab(a+b)$. Since a is positive, we have $a^2-ab-b^2=0$ or if we let $t=\frac{a}{b}$, then $t^2-t-1=0$ and

$$t=\frac{1\pm\sqrt{5}}{2}$$

But a/b is positive and hence we have

$$a=c=\left(\frac{1+\sqrt{5}}{2}\right)b$$

Thus (a) need not imply $a=b=c$.
Suppose (b) holds. Suppose that a,b,c are mutually distinct. We may assume that $a=\max\{a,b,c\}$. Thus $a>b, a>c$. Using $a>b$, we get from the first relation, $a^3+b^3<b^3+c^3$ and hence $a^3<c^3$ and $a<c$, a contradiction. Thus a,b,c can not be mutually distinct. Some two of them must be equal. If $a=b$, the equality of the first two expressions yield $a^3+b^3=b^3+c^3$ and

hence $a = c$. Similarly, if $b = c$ one can deduce that $a = b$ as well or if $c = a$, it follows that $b = c$. Thus $a = b = c$ when (b) holds.

3. Let $\mathbb{N}$ denote the set of all natural numbers. Define a function $T : \mathbb{N} \to \mathbb{N}$ by $T(2k) = k$ and $T(2k+1) = 2k+2$ for all k. We write $T^2(n) = T(T(n))$ and more generally, $T^k(n) = T^{k-1}(T(n))$.

   (a) Show that for each $n \in \mathbb{N}$, there exists a k such that $T^k(n) = 1$.

   (b) For $k \in \mathbb{N}$, let c_k denote the number of elements in the set $\{n : T^k(n) = 1\}$. Prove that $c_{k+2} = c_{k+1} + c_k$ for all $k \geq 1$.

Solution

(a) For $n = 1$, we have $T(1) = 2$ and $T^2(1) = T(2) = 1$. Suppose that $n > 1$. If n is even, then $T(n) = n/2$. If $n/2$ is even, then

$$T^2(n) = T(n/2) = n/4 \leq n - 1$$

when $n > 1$. Again, if $n/2$ is odd, then

$$T^2(n) = T(n/2) = n/2 + 1 \leq n - 1$$

when $n > 3$. On the other hand, if n is odd and $n \geq 3$, $T(n) = n+1$ and $T^2(n) = (n+1)/2$. Again we see that $(n+1)/2 \leq (n-1)$. Thus in all cases, $T^2(n) \leq n - 1$ and in at most $2(n-1)$ steps, T sends n to 1.

(b) Let $n \in \mathbb{N}$ and let $k \in \mathbb{N}$ be such that $T^k(n) = 1$. If n is odd, then

$$T^k(n) = T^{k-1}(T(n)) = T^{k-1}(n+1)$$

and $n+1$ is even.
If n is even and is of the form $4d+2$, then

$$1 = T^k(4d+2) = T^{k-1}(2d+1)$$

Here, $2d+1 = n/2$ is odd.
Thus when n is odd or of the form $4d+2$, each solution of $T^{k-1}(m) = 1$ produces exactly one solution of $T^k(n) = 1$. If $n = 4d$ and $T^k(n) = 1$, then

$$1 = T^k(4d) = T^{k-1}(2d) = T^{k-2}(d)$$

and hence each solution of $T^{k-2}(m) = 1$ produces exactly one solution of $T^k(n) = 1$.
Thus the number of solutions of $T^k(n) = 1$ is equal to the sum of the number of solutions of $T^{k-1}(m) = 1$ and $T^{k-2}(l) = 1$, for $k > 2$ and consequently $c_{k+2} = c_{k+1} + c_k$.
We also observe that 2 is the only number that goes to 1 in one step and 4 is the only number that goes to 1 in two steps. Thus $c_1 = 1$ and $c_2 = 2$. It follows that $c_n = f_{n+1}$, the n-th Fibonacci number.

4. Suppose 2016 points of the circumference of a circle are coloured red and the remaining points are coloured blue. Given any natural number $n \geq 3$, prove that there is a regular n-sided polygon all of whose vertices are blue.
Solution Consider a regular $2017 \times n$ polygon on the circle. Let its vertices be $A_1, A_2, \ldots, A_{2017n}$. For each j, $1 \leq j \leq 2017$, consider the set of points

$$S_j = \{A_k : k = j \mod 2017\}$$

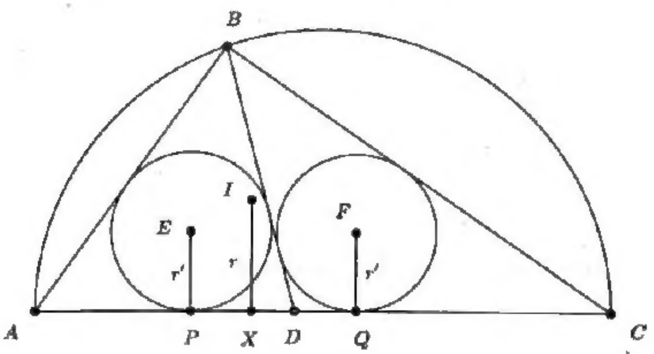
Each S_j is a regular n-gon. Thus we obtain 2017 regular n-gons. Since there are only 2016 red points, by pigeon hole principle, there must be some n-gon among these

2017 that does not contain any red point. That polygon is a blue n-gon.

5. Let ABC be a right angled triangle with $\angle B = 90°$. Let D be a point on AC such that the in-radii of the triangles ABD and CBD are equal. If this common value is r' and if r is the inradius of the triangle ABC, show that

$$\frac{1}{r'} = \frac{1}{r} + \frac{1}{BD}$$

Solution Let E, F be the incenters of the triangles



ABD and BCD respectively. Let the incircles of the triangles ABD, BCD and ABC touch AC and P, Q and X respectively. We have

$$\frac{r'}{r} = \frac{AP}{AX} = \frac{CQ}{CX} = \frac{AP + CQ}{AC}$$

If s_1, s_2 are the semi-perimeters of the triangles ABD and BCD respectively, then $AP = s_1 - BD$ and $CQ = s_2 - BD$. Thus

$$\frac{r'}{r} = \frac{s_1 + s_2 - 2BD}{AC}$$

But

$$\begin{aligned} s_1 + s_2 &= (c + AB + BD + BD + DC + a)/2 \\ &= (a + b + c + 2BD)/2 = s + BD \end{aligned}$$

Thus

$$\frac{r'}{r} = \frac{s - BD}{b} \tag{1}$$

Let us denote the area of a triangle XYZ by $[XYZ]$. We have

$$r' = \frac{[ABD]}{s_1} = \frac{[BCD]}{s_2} = \frac{[ABD] + [BCD]}{s_1 + s_2} = \frac{rs}{s + BD}$$

and hence

$$\frac{r'}{r} = \frac{s}{s + BD} \tag{2}$$

From (1) and (2), we get

$$\frac{s - BD}{b} = \frac{s}{s + BD} \Rightarrow s(s - b) = BD^2$$

But since $\angle B = 90^\circ$, $s - b = r$ and hence $BD^2 = rs = [ABC]$.

Now,

$$\begin{aligned} \frac{1}{r'} &= \frac{s_1}{[ABD]} = \frac{s_2}{[BCD]} \\ &= \frac{s_1 + s_2}{[ABC]} = \frac{s + BD}{[ABC]} = \frac{s + BD}{rs} \\ &= \frac{1}{r} + \frac{BD}{BD^2} \quad \text{since } [ABC] = BD^2 \\ &= \frac{1}{r} + \frac{1}{BD} \end{aligned}$$

This completes the proof.

6. Consider a non-constant arithmetic progression

$$a_1, a_2, \ldots, a_n, \ldots$$

Suppose there exist relatively prime positive integers $p > 1$ and $q > 1$ such that $a_1^2, a_{p+1}^2, a_{q+1}^2$ are also terms of the same arithmetic progression. Prove that the terms of the arithmetic progression are all integers.

Solution Let $a_1 = a$. We have

$$a^2 = a + kd, (a + pd)^2 = a + ld, (a + qd)^2 = a + md \quad (1)$$

and hence

$$a + ld = a^2 + 2apd + p^2d^2 = a + kd + 2apd + p^2d^2 \quad (2)$$

Since the arithmetic progression is non-constant, $d \neq 0$. From (2), we get $2ap + p^2d = l - k$. Similarly, we get $2qa + q^2d = m - k$. Since p, q are relatively prime, we have $p^2q - pq^2 \neq 0$. Solving for a, d, we get

$$a = \frac{p^2(m-k) - q^2(l-k)}{2(p^2q - pq^2)} \quad (3)$$

$$d = \frac{q(l-k) - p(m-k)}{(p^2q - pq^2)} \quad (4)$$

Thus a, d are rational. Also

$$p^2a^2 = p^2a + kp^2d$$

. But $p^2d = l - k - 2pa$ and hence

$$p^2a^2 = p^2a + k(l - k - 2pa) = (p - 2k)pa + k(l - k)$$

Thus pa satisfies the equation

$$x^2 - (p - 2k)x - k(l - k) = 0$$

Since a is rational, pa is rational and since the coefficient of x^2 is 1, it follows that pa is an integer. Similarly,

one can prove that qa is an integer. Since $\gcd(p, q) = 1$, there exist integers u, v such that $pu + qv = 1$ and hence $a = pau + qav$. Thus a is an integer.
Now, $p^2d = l - k - 2pa$ and it follows that p^2d is an integer. Similarly, q^2d is also an integer. Again, since $\gcd(p^2, q^2) = 1$, applying a similar argument as above, it follows that d is an integer. Thus all the terms of the arithmetic progression are integers.